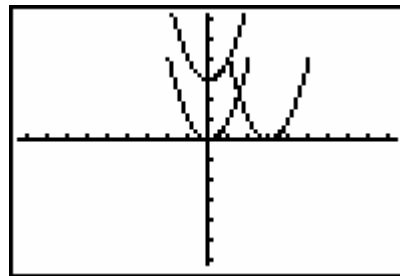
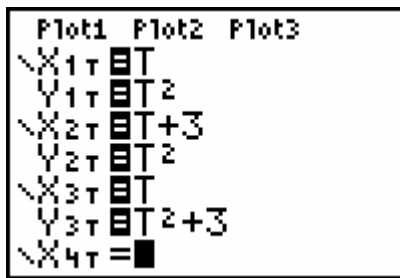


Parametric Representation throughout Pre-Calculus  
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If you look up parametric equations in the index of most Pre-Calculus books, you will probably see one reference located deep in the middle of the chapter on vectors. With the use of technology, however, parametric equations can be an integral part of most of the Pre-Calculus curriculum. I hope to share a few ideas of where I use parametric equations in my own classes.

### Functions and Transformations

I introduce parametric equations in the first unit taught in our Pre-Calculus curriculum, immediately after teaching the definition of a function. The use of parametric equations greatly enhances the understanding of domain and range. I start with the students doing explorations on domain and range, using a prescribed T-min and T-max. I use this activity not only to re-enforce the concepts of domain and range but I also use the T-step to help the students develop an understanding of the difference between a continuous function and a discrete function. It is also a great reintroduction to translations. For example look at the following pairs of graphs, graphed using a T range of -2 to 2.

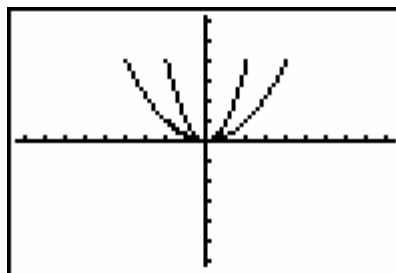


Finding the functional notation for each of these curves, we can relate functional transformation concepts to equations defined parametrically. This can be especially helpful when trying to distinguish between the transformations  $y = af(x)$  and  $y = f(bx)$ . By using the built in domain and range restrictions that parametric equations represent, it is easier to show that although for certain functions transformations may have the same graph, they are affecting the graph in “different” ways.

```

Plot1 Plot2 Plot3
X1T 0T
Y1T 0T^2
X2T 0T^2
Y2T 0T^2
X3T 0T
Y3T 0.25T^2
X4T =

```



### Inverses

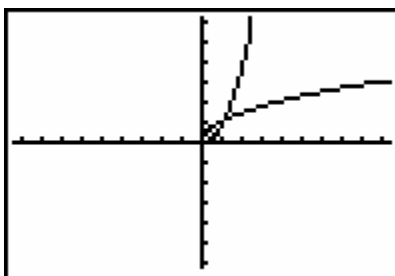
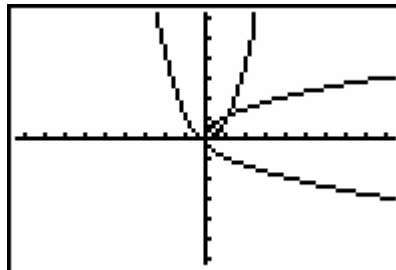
I would feel lost trying to teach the concept of the inverse of a function without using parametric representation. After starting with functional representations of one-to-one functions I move to using parametric equations for both one-to-one and non one-to-one functions. The idea of “flipping” the x and y equations to create the inverse parametrically makes sense to the students. I let the students discover the idea of restricting a domain of a function so that it has a functional inverse, and what makes a “good” restriction. This is especially important when I cover the inverses of trigonometric functions later in the year.

For the two graphs below, the first is graphed with a T range of -4 to 4, the second time with a T range of 0 to 4

```

Plot1 Plot2 Plot3
X1T 0T
Y1T 0T^2
X2T 0T^2
Y2T 0T
X3T =
Y3T =
X4T =

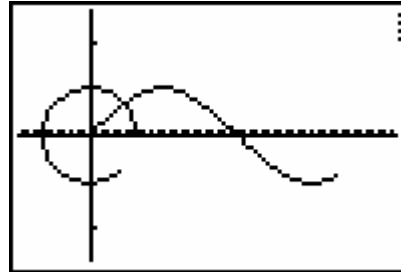
```



## Introduction to Trigonometry

After defining the x and y coordinates of the unit circle in terms of sine and cosine, it is easy to develop the idea of a parametrically defined unit circle. It is nice to show the “unwrapping” of the circle to create the sine graph. For the example shown, I used the calculator in radians mode with a window of  $[0, 2\pi]_{\pi/24} \times [-1.5, 6.5]_{\pi/24} \times [-2.67, 2.67]_1$

```
Plot1 Plot2 Plot3
X1T cos(T)
Y1T sin(T)
X2T T
Y2T sin(T)
X3T =
Y3T =
X4T =
```



## Conic Sections

The Pythagorean trigonometric identities allow for easy parametric representation of ellipses and hyperbolas. Parabolas are most easily represented without the use of trigonometry.

### Ellipses

A comparison of the Pythagorean identity:  $\cos^2 t + \sin^2 t = 1$

and a standard form for the equation of an ellipse:  $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

allows for two simple substitutions:  $\cos^2 t = \frac{(x-h)^2}{a^2}$  and  $\sin^2 t = \frac{(y-k)^2}{b^2}$

Solving these two equations for x and y yields a pair of parametric equations:

$$x = a \cos t + h$$

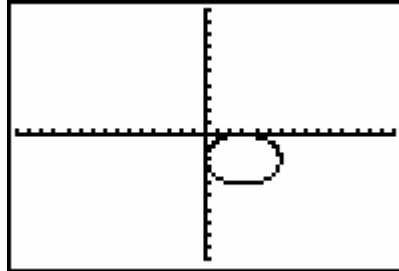
$$y = b \sin t + k$$

A specific example; to graph  $\frac{(x-3)^2}{9} + \frac{(y+2)^2}{4} = 1$  on the TI-83, one would put the calculator in parametric mode. The choice of degree or radian mode is one of personal choice. For the first examples I would do in my class I would use degree mode. Then with a window of  $[0, 360]_5 \times [-15, 15]_1 \times [-10, 10]_1$ , the appropriate equations and graphs are shown:

```

Plot1 Plot2 Plot3
X1T 3cos(T)+3
Y1T 2sin(T)-2
X2T =
Y2T =
X3T =
Y3T =
X4T =

```



A few personal comments are important at this point :

I chose substitutions I did to reinforce the use of x and y coordinates of a unit circle to represent sine and cosine respectively. In using this method I am de-emphasizing the idea that “a” corresponds to the major axis, etc. I focus on the idea that “a” is a stretch in the x equation and therefore a horizontal stretch. Likewise, “b” is a vertical stretch. I’d just as soon not use the letters “a” and “b” at all, but focus on the major axis being the axis with the “largest” stretch.

Some students see a contradiction in the transformation in parametric representation when compared to the Cartesian representation, By re-writing  $x = 3\cos t + 3$  in the form  $x - 3 = 3\cos t$ , I try to show that there is really no contradiction.

## Hyperbolas

By using the Pythagorean identity:  $\sec^2 t - \tan^2 t = 1$

and a standard form for a hyperbola:  $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

One can derive the following pairs of parametric equations to represent hyperbolas:

$$x = a \sec t + h$$

$$y = b \tan t + k$$

(horizontal transverse axis)

$$x = b \tan t + h$$

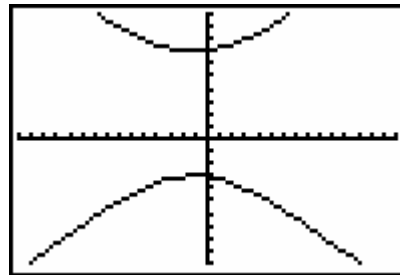
$$y = a \sec t + k$$

(vertical transverse axis)

In a hyperbola, unlike an ellipse, it makes a difference which trigonometric function corresponds with which variable. Therefore using the same window settings as before to

graph the hyperbola  $\frac{(y-2)^2}{25} - \frac{(x+1)^2}{36} = 1$  on the TI-83.

```
Plot1 Plot2 Plot3
X1T 6tan(T)-1
Y1T 5(cos(T)-1+
2
X2T =
Y2T =
X3T =
Y3T =
```



## Parabolas

Parabolas are most easily graphed parametrically without the use of trigonometric functions. All non-rotated parabolas can either be written in the form  $y = f(x)$  or  $x = f(y)$ . Parametrically, parabolas that can be written  $y = f(x)$  can be graphed using  $x = t$  and  $y = f(t)$ , likewise parabolas that can be represented as  $x = f(y)$  can be graphed parametrically using  $x = f(t)$  and  $y = t$ . In this case the T step of the window must be adjusted to include negative values for t or the entire parabola will not appear.

## Extensions

Some possible ideas for extension include trying to develop a trigonometric method of graphing parabolas parametrically, and extending the use of parametric graphing to graph rotated conic equations.

## Vectors

Of course, most traditional work with parametric equations has been related to work with vectors. The use of the TI-83 allows for the exploration of two dimensional motion problems, especially those related with projectile motion. By using the parametric equations:

$$x = (v_o \cos \theta)t$$
$$y = \frac{1}{2}at^2 + (v_o \sin \theta)t + s_o$$

(Where  $v_o$  is initial velocity,  $\theta$  is the angle of projection,  $a$  is acceleration due to gravity and  $s_o$  is initial height at projection.)

One can explore the effects of changing the various parameters. Many calculator games have been developed that use these ideas in situations such as throwing a basketball or javelin, or hitting a baseball.