Tropical Mathematics

An Interesting and Useful Variant of Ordinary Arithmetic

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- Starts with a new arithmetic
- Includes polynomials, curves, higher algebra
- Useful in combinatorics, algebraic geometry
- Useful in genetics
- It is fun to do math in a different setting

Why Tropical Mathematics?

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- The name simply reflects how a few Frenchmen view Brazil

Tropical Arithmetic

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- Ordinary arithmetic
  - Real numbers, addition ($+$) and multiplication ($\times$)
- Tropical arithmetic
  - Real numbers plus infinity, denoted by ∞
  - Tropical addition ($\oplus$)
  - Tropical multiplication ($\otimes$)

Tropical Addition

$a \oplus b =$ the minimum of a and b.

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Examples:

$$3 \oplus 5 = 3, \qquad 3 \oplus (-5) = -5$$

$$12 \oplus 0 = 0, \qquad 0 \oplus (-3) = -3$$

$$12 \oplus \infty = 12, \qquad \infty \oplus x = x$$

Tropical Addition Table

$\oplus$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2
3	1	2	3	3	3	3	3
4	1	2	3	4	4	4	4
5	1	2	3	4	5	5	5
6	1	2	3	4	5	6	6
7	1	2	3	4	5	6	7

Differences

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- The equation $3 \oplus x = 5$ has no solution.

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- The equation $3 \oplus x = 1$ has the solution $x = 1$.
- The equation $3 \oplus x = 5$ has no solution.
- The equation $a \oplus x = \infty$ has no solution if $a \neq \infty$.

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$$3 \otimes 5 = 8, \qquad 3 \otimes (-5) = -2,$$
$$(-1) \otimes 3 = 2, \qquad \infty \otimes (-1) = \infty.$$

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$$3 \otimes 5 = 8, \qquad 3 \otimes (-5) = -2,$$
$$(-1) \otimes 3 = 2, \qquad \infty \otimes (-1) = \infty.$$

The multipicative unit is 0.

$$0 \otimes x = x \otimes 0 = x \quad \text{for all } x.$$

Tropical Multiplication Table

$\otimes$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	7
2	2	3	4	5	6	7	8
3	3	4	5	6	7	8	9
4	4	5	6	7	8	9	10
5	5	6	7	8	9	10	11
6	6	7	8	9	10	11	12

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- The distributive law is valid
- $(x \oplus y)^3 = x^3 \oplus y^3$

Linear Functions

- The graph of $y = 5$ is a straight line with slope 0.
- The graph of $y = 3 \otimes x$ is a straight line with slope 1.
- The graph of $y = 3 \otimes x \oplus 5 = \min\{3 \otimes x, 5\}$ is a crooked line.
- $y = 3 \otimes x \oplus 5 = 3 \otimes (x \oplus 2)$
- $x = 2$ is where the graph bends.

Monomials

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$$x^2 = x \otimes x = x + x = 2x$$

$$x^3 = x \otimes x \otimes x = 3x$$

$$x^p = p \times x$$

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$$x^p = p \times x$$

Monomials are linear functions with integer coefficients.

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The graph is a line with slope 2.

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$$3 \otimes x^2 = 3 + (2x)$$

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The graph is a line with slope 3.
The exponent of a monomial is the slope of its graph.

Polynomials

Example 1:

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$$p(x) = 2 \otimes x^2 \oplus x \oplus 5$$
$$= \min\{2x + 2, x, 5\}$$

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$$= \min\{2x + 2, x, 5\}$$

The graph is a twice bent line.
The graph bends at $x = -2$ and $x = 5$.
We can show that $p(x) = 2 \otimes [x \oplus (-2)] \otimes [x \oplus 5]$.

Polynomials

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$$p(x) = x^2 \oplus 3 \otimes x \oplus 2$$
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Example 2:

$$p(x) = x^2 \oplus 3 \otimes x \oplus 2$$
$$= \min\{2x, x + 3, 2\}$$

The graph is a once bent line.
The graph bends at $x = 1$.
We can show that $p(x) = (x \oplus 1)^2$.

Factorization of Polynomials

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- Our two example polynomials factor into linear factors.
- The factors have the form $x \oplus a$, where a is a bend point for the graph.
- Any tropical polynomial can be expressed in one and only one way as the product of linear factors.
- Thus the Fundamental Theorem of Algebra remains true in tropical mathematics.

Fundamental Theorem of Algebra

- All polynomials factor into powers of linear factors.

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- All polynomials factor into powers of linear factors.
- A factor of the form $(x \oplus a)^p$ means that the graph bends at $x = a$, with the slope decreasing by p.

Polynomials in Two Variables

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- A polynomial represents the minimum of one or more linear functions. Example: $p(x, y) = x \oplus y \oplus 1 = \min\{x, y, 1\}$
- The bend points of the graph occur where two or more of the linear functions agree.

Curves

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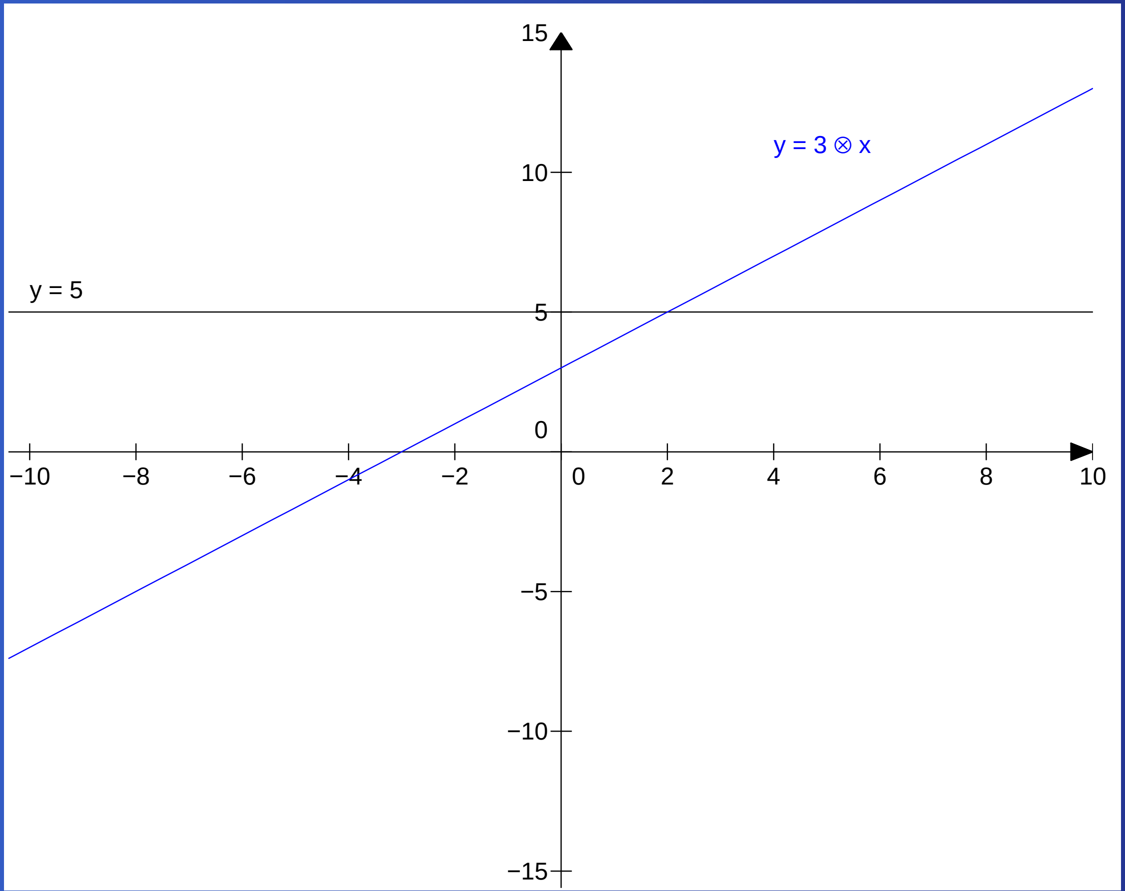
- In tropical math, the zero set is replaced with the bend set — a tropical curve.
- Examples:

1. $p(x, y) = x \oplus y \oplus 1 = \min\{x, y, 1\}$
2. $p(x, y) = x^2 \oplus y^2 \oplus 4 = \min\{2x, 2y, 4\}$
3. $p(x, y) = x^2 \oplus y^2 \oplus x \oplus 4 = \min\{2x, 2y, x, 4\}$

The End

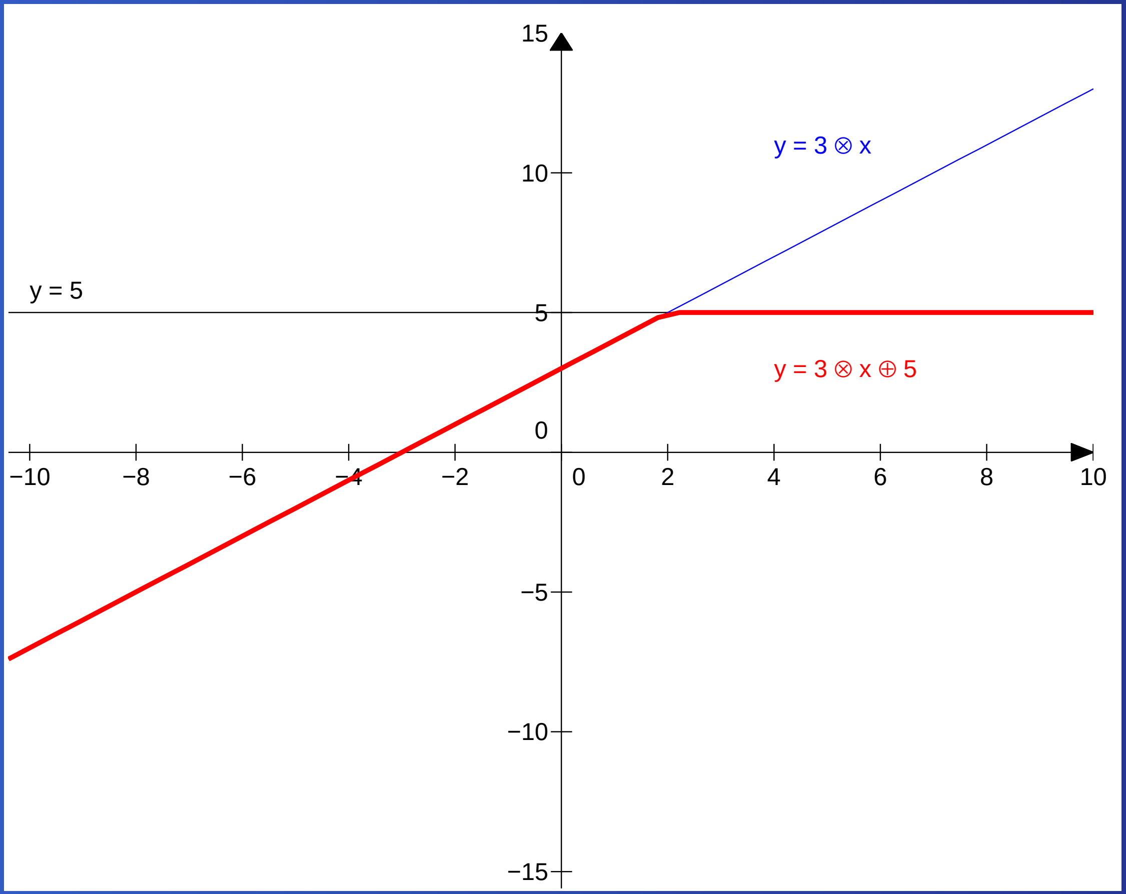
Graphs of y $= 5$ & y $= 3 \otimes x$



Return

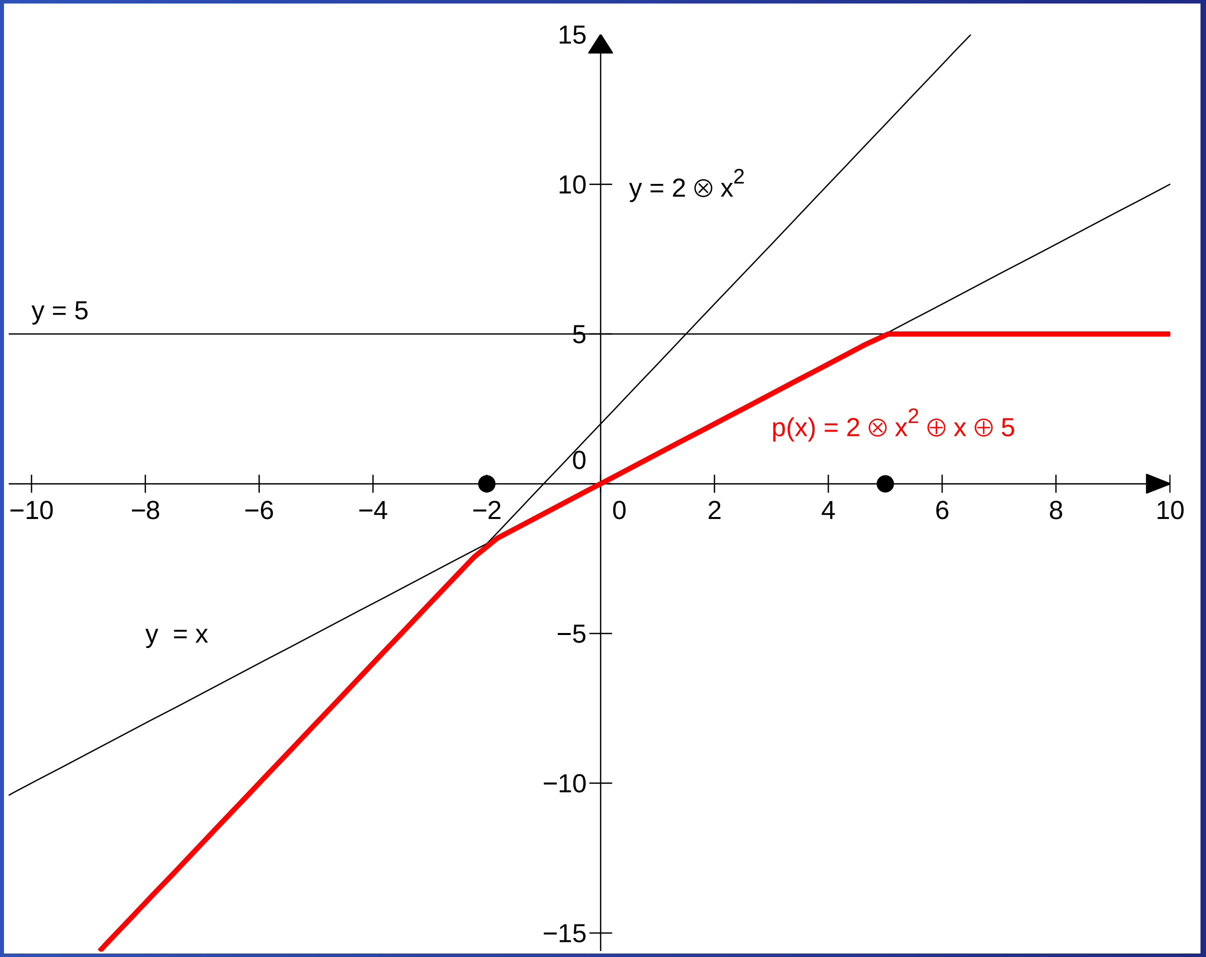
$$y = 3 \otimes x \oplus 5$$

Graph of y $= 3 \otimes x \oplus 5$



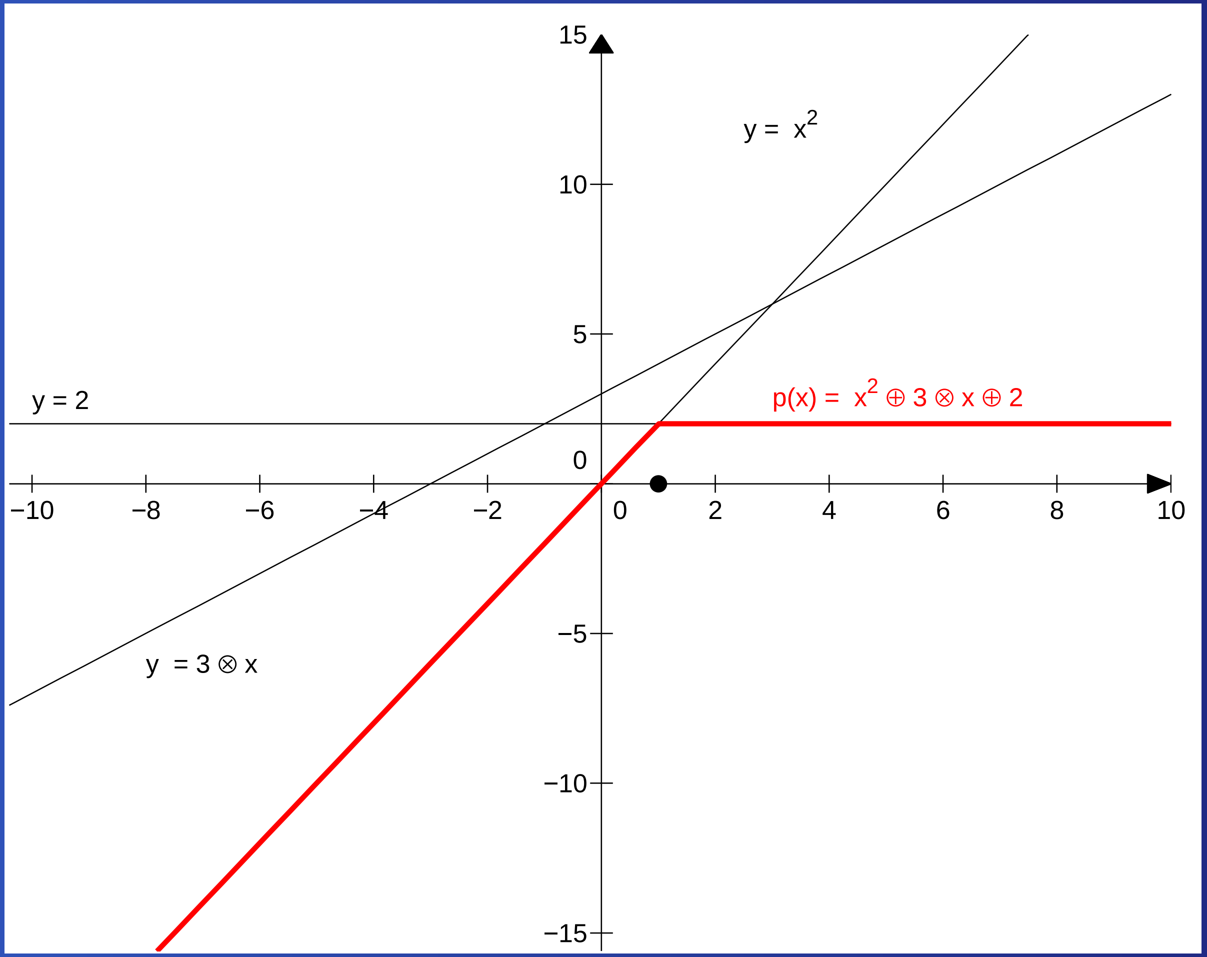
Return

Graph of y $= 2 \otimes x^2 \oplus x \oplus 5$



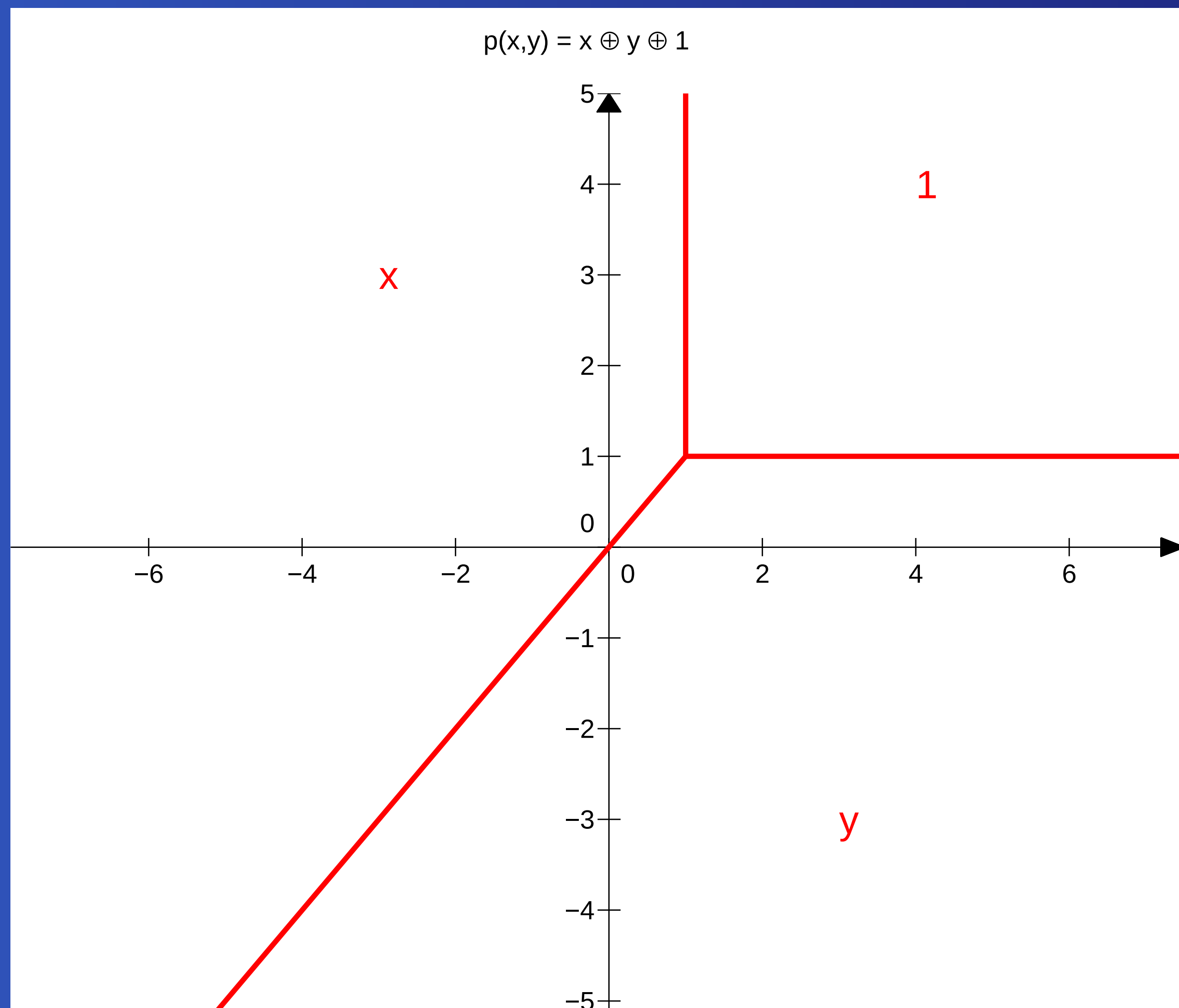
Return

Graph of y $= x^2 \oplus 3 \otimes x \oplus 2$



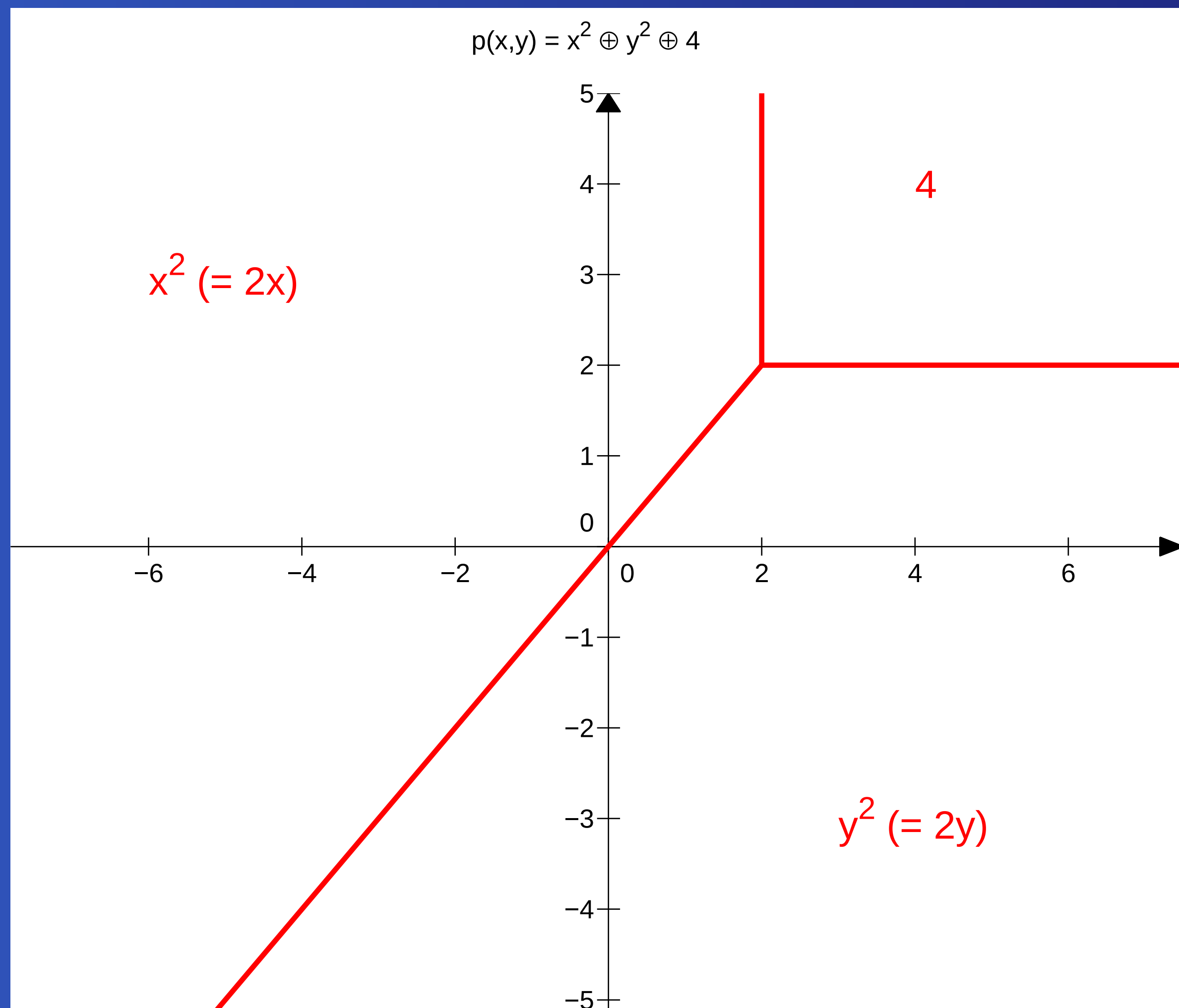
Return

Bend set of $\mathrm{p}(x, y) = x \oplus y \oplus 1$



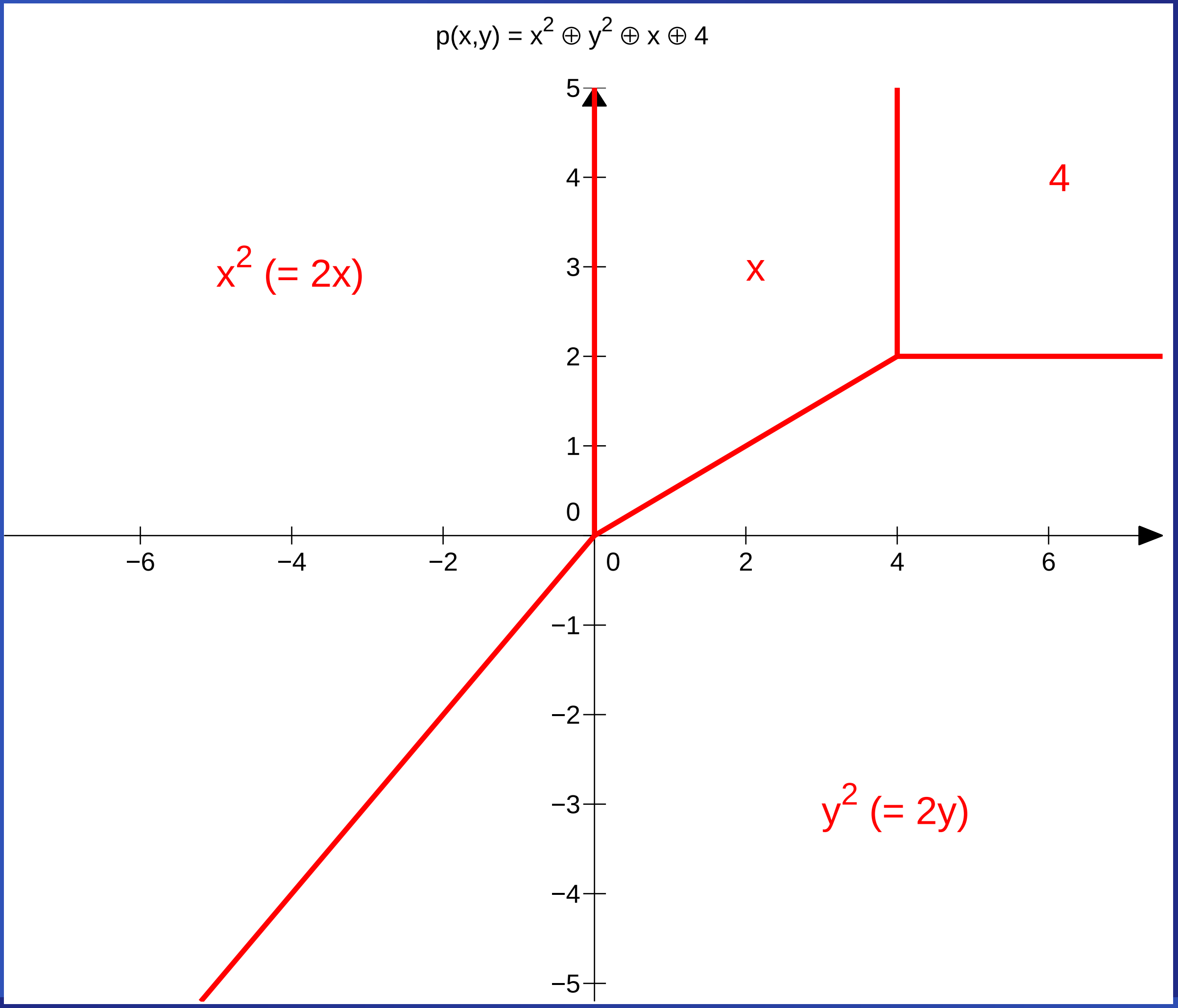
Return

Bend set of $\mathrm{p}(x, y) = x^2 \oplus y^2 \oplus 4$



Return

Bend set of $x^2 \oplus y^2 \oplus x \oplus 4$



Return