

Unit 2 - Triangles

Equilateral Triangles

Overview: In this activity participants discover properties of equilateral triangles using properties of symmetry.

Objective: **TE_xES Mathematics Competencies**
III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.012.E. The beginning teacher describes and justifies geometric constructions made using compass and straightedge, reflection devices, and other appropriate technologies.
III.013.A. The beginning teacher analyzes the properties of polygons and their components.
III.013.B. The beginning teacher analyzes the properties of circles and the lines that intersect them.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).

Geometry TEKS

b.1.A. The student develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems.
b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.4. The student uses a variety of representations to describe geometric relationships and solve problems.
c.3. The student identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.

e.2.C. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of circles and the lines that intersect them.

Background: Participants need a basic understanding of reflections, lines of reflection, lines of symmetry, and properties of similar figures.

Materials: patty paper, straightedge compass, easel paper, colored markers

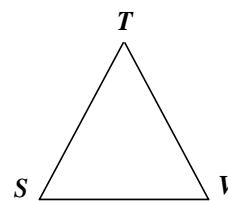
New Terms: altitude, central angle, circumscribed circle, concentric circle, inscribed angle, inscribed circle, line of reflection or line of symmetry, median, perpendicular bisector

Procedures:

Distribute the activity sheet, easel paper, and markers to participants. Have them work for about 45 minutes, answering the questions individually on their own paper. Allow discussion with others when needed. Participants record their responses to 12 on a sheet of easel paper. During whole group discussion, ask one participant from each group to summarize the properties determined by his/her group. Ask other participants to add any properties omitted by earlier groups. Be careful to list all of the properties that appear below, including the connections to special right triangles, similar figures and circles from 10 and 11.

Complete 1-12 to explore the properties of equilateral triangles.

1. Carefully construct a large equilateral triangle on patty paper using a straightedge and compass. Label $\triangle STV$.

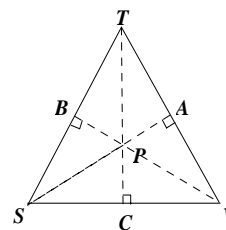


2. List properties of equilateral triangles and mark the triangle to indicate the identified properties. Explain how you know these properties from the constructed triangle.

Participants may write that equilateral triangles have equal side lengths and equal angle measures. These properties can be verified by folding the angles on top of each other.

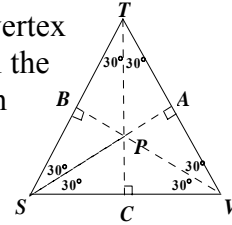
3. Fold, draw or construct the lines of symmetry/reflection on the patty paper triangle. Label the intersection of the lines of symmetry P .

4. Label the intersection of the line of symmetry from vertex S to \overline{TV} as A , the intersection of the line of symmetry from vertex V to \overline{TS} as B , and the intersection of the line of symmetry from vertex T to \overline{SV} as C .



5. Describe the relationship between the lines of symmetry and the vertex angles of the triangle. Mark these properties on the triangle. Label the measures of the angles created by the lines of symmetry. Based on these properties, what is another name for the lines of symmetry?

The lines of symmetry can be called angle bisectors because the segments bisect the vertex angles, forming two 30° angles.

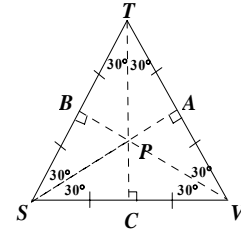


6. Describe the relationship between the lines of symmetry and the sides of the triangle. Mark the relationships on the triangle.

The lines of symmetry bisect the sides of the triangle.

$$\overline{TB} \cong \overline{SB}; \overline{SC} \cong \overline{VC}; \overline{TA} \cong \overline{VA}.$$

The lines of symmetry are perpendicular to the sides of the triangle. $\overline{VB} \perp \overline{TS}$; $\overline{SA} \perp \overline{TV}$; $\overline{TC} \perp \overline{SV}$.



7. Using the information now marked on the triangle, what are three other names which can be used to describe the segments lying on the lines of symmetry?

Because the three lines of symmetry bisect the sides of the triangle at right angles, the segments lying on the lines of symmetry are also medians, altitudes and perpendicular bisectors. A median of a triangle is a segment connecting a vertex to the midpoint of the opposite side of the triangle. An altitude is a segment from a vertex, perpendicular to the opposite side. A perpendicular bisector is a segment that bisects another segment at a right angle.

8. Use a ruler or patty paper to compare the lengths of the medians.

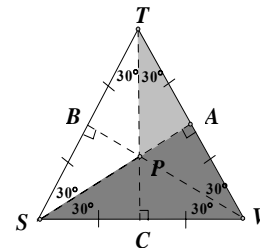
The medians, which lie on the lines of symmetry, are congruent.

9. Use a ruler or patty paper to compare the lengths of the longer and shorter segments that make up a median, for example, \overline{TP} and \overline{PC} . Find the ratio of their lengths. Why is the ratio the same for all three medians?

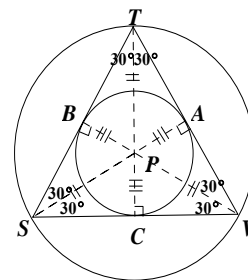
The ratio of the longer to the shorter segment is 2:1. For example, $TP = 2 \cdot PC$. This holds true for all of the medians because the symmetry properties ensure congruence for all of the longer segments and all of the shorter segments.

10. Compare the six smaller right triangles and the six larger right triangles formed by the medians. Describe the relationship between the larger right triangles and the smaller right triangles.

All of the right triangles have angle measures of 30°, 60°, and 90°. We call these 30°-60°-90° triangles. The larger and smaller right triangles are similar. Similar shapes have congruent angles, and proportional side lengths. In the large 30°-60°-90° triangle, the length of the hypotenuse, a side of the equilateral triangle, is twice the length of the short side, formed by the bisection of a side of the equilateral triangle. The relationship must exist for both the small and large triangles. This explains the 2:1 ratio for the medians.



11. Using a compass, draw a circle inscribed in $\triangle TSV$, radius \overline{PB} .
 Draw a second circle circumscribed about $\triangle TSV$, radius \overline{PT} .
 Record the properties of an inscribed circle and a circumscribed circle for an equilateral triangle.



- *The length of the radius of the circumscribed circle is twice the length of the radius of the inscribed circle in an equilateral triangle.*
- *The two circles have the same center; they are concentric circles.*
- *$\angle SPV \cong \angle TPV \cong \angle TPS$. These are central angles for the circumscribed circle, because their vertices are located at the center of the circle, and their sides are radii of the circle. $\angle STV$, $\angle TSV$, and $\angle TVS$ are inscribed angles, intercepted by the same chords, \overline{SV} , \overline{TV} , and \overline{TS} , as the central angles, in that order. Note that the measures of these central angles are 120° , while the measures of the corresponding inscribed angles are 60° . The measure of the central angle is equal to twice the measure of the inscribed angle intercepted by the same chord.*

12. On a sheet of easel paper, construct and label an equilateral triangle with the lines of symmetry, and the inscribed and circumscribed circles. List all of the properties of equilateral triangles.
- *Equilateral triangles have congruent sides and congruent angles.*
 - *Equilateral triangles have three lines of reflectional symmetry.*
 - *The vertex angles measure 60° .*
 - *The three lines of symmetry bisect the vertex angles.*
 - *The three segments that lie on the lines of symmetry are angle bisectors, altitudes, medians and perpendicular bisectors.*
 - *The lines of symmetry divide the triangle into six larger and six smaller 30° - 60° - 90° triangles.*
 - *The larger and smaller right triangles are similar.*
 - *Within all 30° - 60° - 90° triangles, the length of the hypotenuse is twice the length of the short leg.*
 - *The inscribed and circumscribed circles are concentric circles centered at the intersection of the lines of symmetry.*
 - *The radius of the circumscribed circle is twice as long as the radius of the inscribed circle.*
 - *The measure of a central angle is two times the measure of the inscribed angle intercepted by the same chord.*

At the end of the activity, facilitate a whole group discussion about the van Hiele levels used throughout the activity on equilateral triangles. Participants making observations in 1 – 4 are performing on the Visual Level, using quick observation. With 5 – 12, the questions move participants to the Descriptive Level because an exhaustive list of properties is developed by further observation and measurement. Although the relationships between different properties are explored in these problems, the questions revolve around one triangle, the equilateral triangle.

Equilateral Triangles

Complete 1 – 12 to explore the properties of equilateral triangles.

1. Carefully construct a large equilateral triangle on patty paper using ruler and compass. Label $\triangle STV$.
2. List properties of equilateral triangles and mark the triangle to indicate the identified properties. Explain how you know these properties from the constructed triangle.
3. Fold, draw or construct the lines of symmetry/reflection on the patty paper triangle. Label the intersection of the lines of symmetry P .
4. Label the intersection of the line of symmetry from vertex S to \overline{TV} as A , the intersection of the line of symmetry from vertex V to \overline{TS} as B , and the intersection of the line of symmetry from vertex T to \overline{SV} as C .
5. Describe the relationship between the lines of symmetry and the vertex angles of the triangle. Mark these properties on the triangle. Label the measures of the angles created by the lines of symmetry. Based on these properties, what is another name for the lines of symmetry?
6. Describe the relationship between the lines of symmetry and the sides of the triangle. Mark the relationships on the triangle.
7. Using the information now marked on the triangle, what are three other names which can be used to describe the segments lying on the lines of symmetry?

8. Use a ruler or patty paper to compare the lengths of the medians.
9. Use a ruler or patty paper to compare the lengths of the longer and shorter segments that make up a median, for example, \overline{TP} and \overline{PC} . Find the ratio of their lengths. Why is the ratio the same for all three medians?
10. Compare the six smaller right triangles and the six larger right triangles formed by the medians. Describe the relationship between the larger right triangles and the smaller right triangles.
11. Using a compass, draw a circle inscribed in ΔTSV with radius \overline{PB} . Draw a second circle circumscribed about ΔTSV with radius \overline{PT} . Record the properties of an inscribed circle and a circumscribed circle for an equilateral triangle.
12. On a sheet of easel paper, construct and label an equilateral triangle with the lines of symmetry and the inscribed and circumscribed circles. List all of the properties of equilateral triangles.

Two Congruent Angles

Overview: Participants construct right, acute, and isosceles obtuse triangles using the radii of congruent circles to determine properties of isosceles triangles. An extension introduces and extends properties of circle segments and angles.

Objective: **TEXES Mathematics Competencies**

III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.

III.013.A. The beginning teacher analyzes the properties of polygons and their components.

III.013.B. The beginning teacher analyzes the properties of circles and the lines that intersect them.

III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).

V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

VI.020.A. The beginning teacher applies research-based theories of learning mathematics to plan appropriate instructional activities for all students.

Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.

b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

b.3.B. The student constructs and justifies statements about geometric figures and their properties.

c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.

e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.

e.2.C. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of circles and the lines that intersect them.

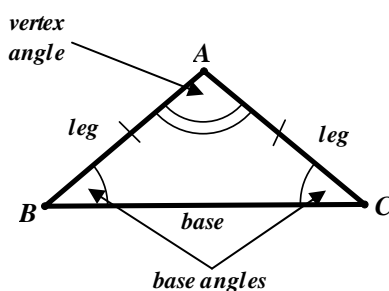
Background: Participants must understand rotational symmetry.

Materials: patty paper, straightedge, protractor, compass, easel paper, colored markers

New Terms:**Procedures:**

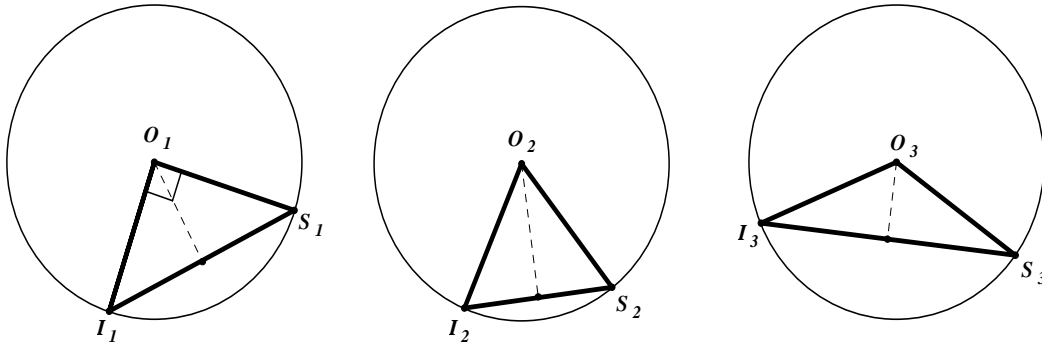
Distribute the activity sheet. Post a large sheet of easel paper headed “Isosceles Triangles.”

1. The figure represents an isosceles triangle. In your group discuss and write down the properties of isosceles triangles that can be observed in the given figure. Be prepared to share these in whole class discussion.



Allow about 5 minutes for group discussion on 1. Ask a participant to record properties on the easel paper, while others share their group's properties. The properties will be critiqued at the end of the activity.

2. You will need three sheets of patty paper.
 - Construct three congruent circles, one on each sheet of patty paper.
 - On the first sheet, label the center O_1 . Draw two radii forming a right angle at the center of the circle. Draw the chord connecting the endpoints of the radii, I_1 and S_1 , forming an isosceles right triangle, $\Delta I_1S_1O_1$.
 - On the second sheet, label the center O_2 . Draw two radii forming an acute angle at the center of the circle. Connect the endpoints of the radii to complete the triangle, an isosceles acute triangle, $\Delta I_2S_2O_2$.
 - On the third sheet, label the center O_3 . Draw two radii forming an obtuse angle at the center of the circle. Complete the triangle, an isosceles obtuse triangle, $\Delta I_3S_3O_3$.



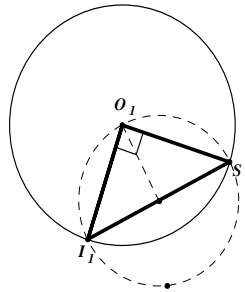
- Find and crease the line(s) of symmetry associated with each triangle. Based on the properties of reflection, determine additional properties of isosceles triangles. In your group, share your findings. Make a group list of properties shared by all isosceles triangles and separate lists for properties unique to isosceles right, isosceles acute and isosceles obtuse triangles.

Allow participants 15-20 minutes to construct the figures and discuss the properties. Then ask participants to provide additional properties to be added to the easel paper poster. After all properties have been posted, critique the properties provided at the beginning of the lesson. Participants justify why the properties are true based on the symmetry properties within these triangles. If participants are unable to generate properties, try to prompt with probing questions rather than providing the list of properties.

Possible property statements (justifications in parentheses):

- Any isosceles triangle has one line of symmetry bisecting the vertex angle and the base. The median to the base lies on the line of symmetry and connects the vertex angle to the midpoint of the base.
- Isosceles triangles have congruent legs (symmetry property).
- Isosceles triangles have congruent base angles (symmetry property).
- The median to the base is also the altitude to the base and the perpendicular bisector of the base (the symmetry line); it creates right angles where it intersects and bisects the base, as well as bisecting the vertex angle (symmetry property).

Isosceles right triangles:



- The median to the base creates two smaller isosceles right triangles (The base angles of the original triangle each measure 45° . The right vertex angle is bisected to form two 45° angles.)

- The median is congruent to the two segments formed by the median on the base. (These are the legs of the smaller isosceles right triangles created by the line of symmetry.)
- The midpoint of the base is the circumcenter of the triangle. (The three segments radiating from this point to the vertices of the triangle are all congruent, forming radii of the circumscribed circle.)
- The length of the base is twice the length of the altitude (the median).

Isosceles acute triangle:

- The length of the base is shorter than two times the length of the altitude.

Isosceles obtuse triangle:

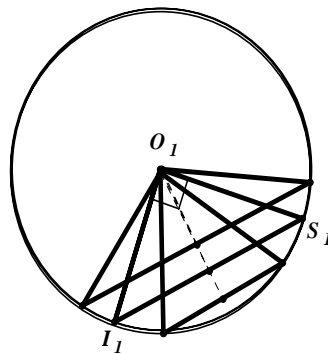
- The length of the base is longer than two times the length of the altitude.

Extension investigations:

3. The bases of the isosceles triangles are chords of congruent circles. The vertex angles are the central angles subtended by the chords. The lengths of the altitudes to the bases are the distances from the chords to the centers of the circles. Examine your figures with this in mind, and in your group make conjectures relating chord length to the distance from the center of the circle.

The shorter the length of the chord, the greater is its distance from the center of the circle. The longer the length of the chord, the shorter is its distance from the center of the circle. As the measure of the central angle increases, the length of the chord increases.

If participants are unable to see the chord and distance to center relationships, encourage them to overlay the centers of the three circles and the lines of symmetry.

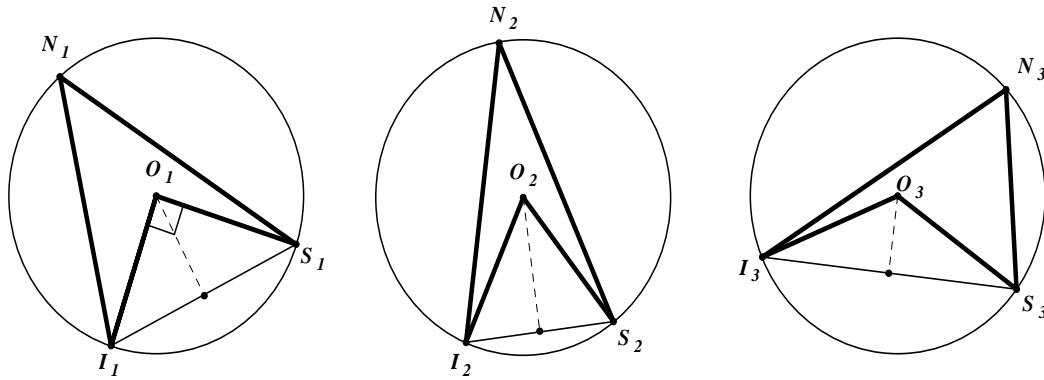


Can the chord continue to grow longer indefinitely?

No, the longest chord is the diameter, which passes through the center of the circle. Its central angle measures 180° .

4. Place points, labeled N_1 , N_2 , or N_3 , respectively on each circle, so that major arcs $\widehat{I_1N_1S_1}$, $\widehat{I_2N_2S_2}$, and $\widehat{I_3N_3S_3}$ are formed. Draw inscribed angles $\angle I_1N_1S_1$, $\angle I_2N_2S_2$, and $\angle I_3N_3S_3$. A conjecture regarding the central angle and the inscribed

angle was made in the activity on equilateral triangles. Use each of your circles to verify the relationship of the central angle to the inscribed angle that intercepts the same arcs.



The measure of the central angle is equal to twice the measure of the inscribed angle intercepted by the same arc.

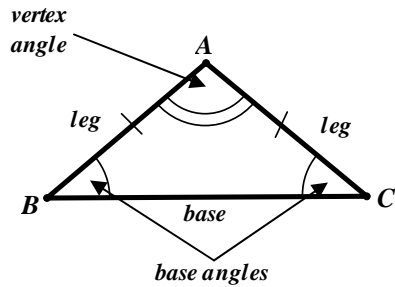
Participants may offer equivalent statements.

At this time an inductive approach, using a protractor or patty paper, is appropriate for the van Hiele Descriptive Level of understanding. Participants may be familiar with the theorem that the measure of the central angle is twice the measure of the inscribed angle intercepted on the same arc. It is important that they understand that the deductive process can usually be followed by those proficient at the Relational Level. This activity guides development of the Descriptive Level, which expects properties to emerge out of inductive approaches. Discuss the different van Hiele levels presented in this activity. In 1 and 2 participants are performing at the Visual Level. Recognition of the figures is all that is needed to do the constructions.

In 2 and 3, the work on the properties moves participants to the Descriptive Level, but approaches the Relational Level in 4. In 2 and 3, the circle is used to describe the properties of the isosceles triangle. In 4, properties of circles are connected to those of isosceles triangles.

Two Congruent Angles

- The figure represents an isosceles triangle. In your group, discuss and write down the properties of isosceles triangles that can be observed in the given figure. Be prepared to share these in whole class discussion.



- You will need three sheets of patty paper.
 - Construct three congruent circles, one on each sheet of patty paper.
 - On the first sheet, label the center O_1 . Draw two radii forming a right angle at the center of the circle. Draw the chord connecting the endpoints of the radii, I_1 and S_1 , forming an isosceles right triangle, $\Delta I_1 S_1 O_1$.
 - On the second sheet, label the center O_2 . Draw two radii forming an acute angle at the center of the circle. Connect the endpoints of the radii to complete the triangle, an isosceles acute triangle, $\Delta I_2 S_2 O_2$.
 - On the third sheet, label the center O_3 . Draw two radii forming an obtuse angle at the center of the circle. Complete the triangle, an isosceles obtuse triangle, $\Delta I_3 S_3 O_3$.
 - Find and crease the line(s) of symmetry associated with each triangle. Based on the properties of reflection, determine additional properties of isosceles triangles. In your group, share your findings. Make a group list of properties shared by all isosceles triangles and separate lists for properties unique to isosceles right, isosceles acute and isosceles obtuse triangles.

Extension investigations:

3. The bases of the isosceles triangles are chords of congruent circles. The vertex angles are the central angles subtended by the chords. The lengths of the altitudes to the bases are the distances from the chords to the centers of the circles. Examine your figures with this in mind, and in your group make conjectures relating chord length and distance from the center of the circle.

4. Place points, labeled N_1 , N_2 , or N_3 , respectively on each circle, so that major arcs $\widehat{I_1N_1S_1}$, $\widehat{I_2N_2S_2}$, and $\widehat{I_3N_3S_3}$ are formed. Draw inscribed angles $\angle I_1N_1S_1$, $\angle I_2N_2S_2$, and $\angle I_3N_3S_3$. A conjecture regarding the central angle and the inscribed angle was made in the activity on equilateral triangles. Use each of your circles to verify the relationship of the central angle to the inscribed angle that intercepts the same arcs.

Scalene Triangles

Overview: Participants determine properties that apply to all scalene triangles.

Objective: **TEXES Mathematics Competencies**

III.012.B. The beginning teacher uses properties of points, lines, planes, angles, lengths, and distances to solve problems.

III.012.E. The beginning teacher describes and justifies geometric constructions made using compass and straightedge, reflection devices, and other appropriate technologies.

III.013.A. The beginning teacher analyzes the properties of polygons and their components.

III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).

Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.

b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

b.3.B. The student constructs and justifies statements about geometric figures and their properties.

e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.

Background: Participants need a knowledge of the properties of isosceles and equilateral triangles.

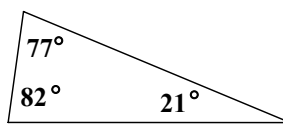
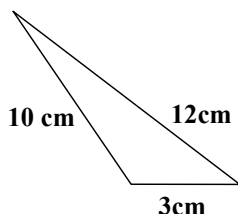
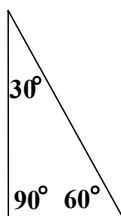
Materials: patty paper, centimeter ruler, compass, protractor

New Terms:

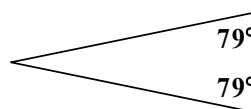
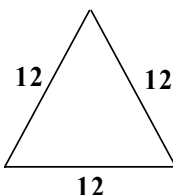
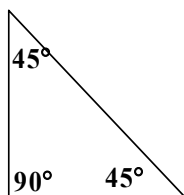
Procedures:

Distribute the activity sheet. Participants answer 1-3 regarding the properties of scalene triangles and triangle inequalities. The following is taken from *Discovering Geometry: an Investigative Approach*, 3rd Edition, ©2003, with permission from Key Curriculum Press.

These triangles are scalene triangles.



These triangles are not scalene triangles.



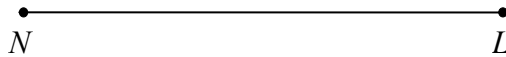
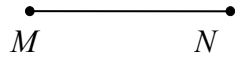
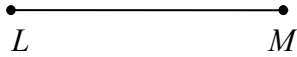
1. What is a scalene triangle?
A scalene triangle is a triangle with no congruent sides and no congruent angles.
2. Name three types of scalene triangles.
Scalene triangles can be scalene right, scalene acute or scalene obtuse.
3. Using the three scalene triangles above, explore and summarize the properties of scalene triangles for symmetry.
No properties result from symmetry, because scalene triangles have no symmetry.
4. Investigate the measures of the angles in triangles and the lengths of the sides opposite those angles. Draw a scalene triangle on your paper. Measure each angle using a protractor. Using a ruler, measure the length of each side in centimeters, rounding to the nearest tenth of a centimeter. Label the measures of the angles and side lengths. What relationship exists among the measures of the angles and the side lengths of the triangle?
The largest angle is opposite the longest side; the smallest angle is opposite the shortest side; the remaining angle is opposite the remaining side.

The following is taken from *Discovering Geometry: an Investigative Approach*, 3rd Edition, ©2003, p 214, with permission from Key Curriculum Press.

5. Investigate geometric inequalities using either patty paper or a compass and straightedge. Complete each construction and compare your results with group members.

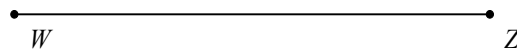
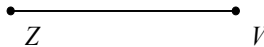
Construct triangles with each set of segments as sides.

a)



These segments do not create a triangle.

b)

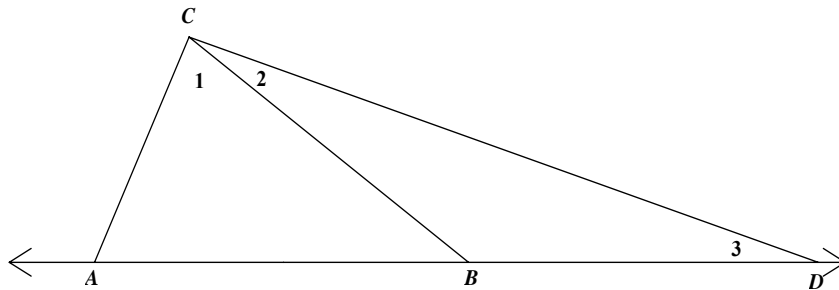


These segments create a triangle.

c) Describe the relationship among the lengths of the segments needed to create a triangle.

The sum of the lengths of any two sides of a triangle must be greater than the length of the third side.

6. In the figure, $BC = BD$.



a. Why is $AD > CA$?

*In $\triangle ACD$, \overline{CA} is opposite $\angle 3$ and \overline{AD} is opposite $\angle ACD$.
 $\angle 2 \cong \angle 3$ ($\triangle BCD$ is an isosceles triangle.)*

$m\angle ACD = m\angle 1 + m\angle 2$ (The whole must be equal to the sum of its parts.)

By substitution, $m\angle ACD = m\angle 1 + m\angle 3$.

Therefore $m\angle ACD > m\angle 3$, and $AD > CA$ (The larger side is opposite the larger angle; the smaller side is opposite the smaller angle.)

- b. Why is $AB + BC > CA$?

$AD = AB + BD$.

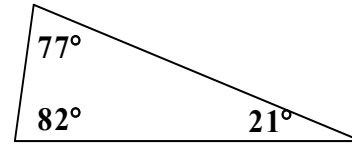
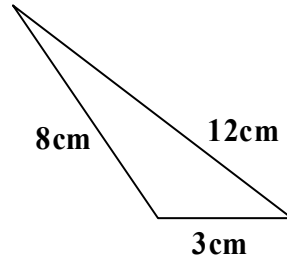
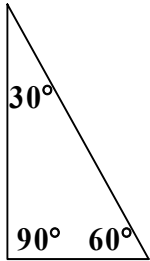
By substitution, $AD = AB + BC$, since $BC = BD$.

Since $AD > CA$, from part a, $AB + BC > CA$.

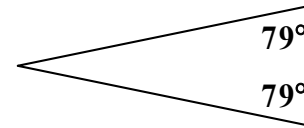
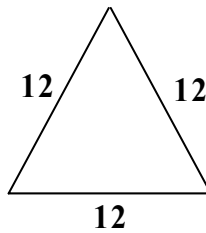
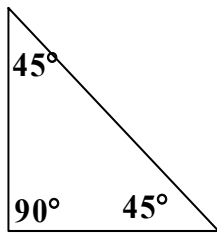
At the end of the activity, discuss the van Hiele levels represented in the activity. In the first three problems, participants perform at the Visual Level, where the language of the concept is clarified. In 3-5 participants determine and describe properties of triangles, at the Descriptive Level. In 6 the steps of the proof are provided, allowing participants to approach the Relational Level. If the proof had been created with only the given information, then participants would have been performing on the Deductive Level.

Scalene Triangles

These triangles are scalene triangles.



These triangles are not scalene triangles.

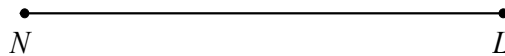
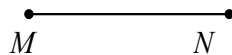
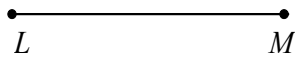


1. What is a scalene triangle?
2. Name three types of scalene triangles.
3. Using the three scalene triangles above, explore and summarize the properties of scalene triangles for symmetry.

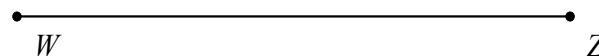
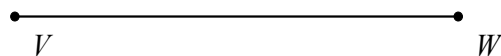
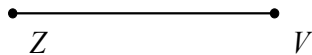
4. Investigate the measures of the angles in triangles and the lengths of the sides opposite those angles. Draw a scalene triangle on your paper. Measure each angle using a protractor. Using a ruler, measure the length of each side in centimeters, rounding to the nearest tenth of a centimeter. Label the measures of the angles and side lengths. What relationship exists among the measures of the angles and the side lengths of the triangle?
5. Investigate geometric inequalities using either patty paper or a compass and straightedge. Complete each construction and compare your results with group members.

Construct triangles with each set of segments as sides.

a)

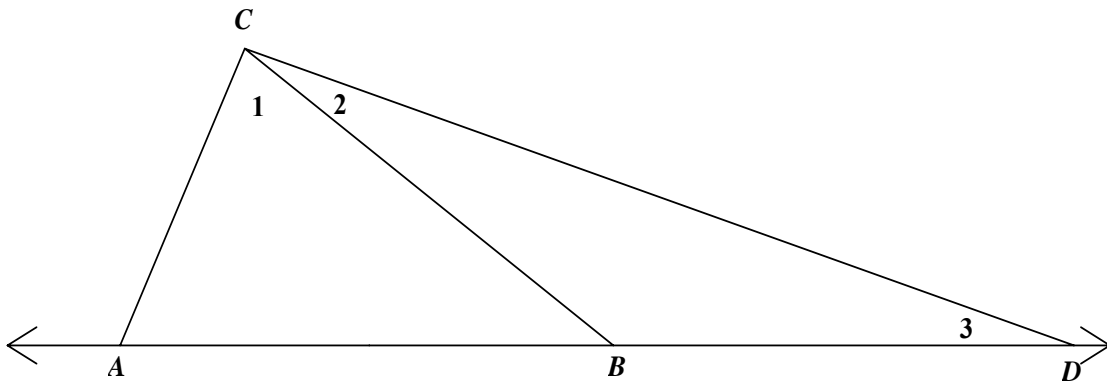


b)



- c) Describe the relationship among the lengths of the segments needed to create a triangle.

6. In the figure, $BC = BD$.



a. Why is $AD > CA$?

b. Why is $AB + BC > CA$?

The Meeting Place

Overview: Participants construct the circumcenter, incenter, centroid and orthocenter in congruent triangles and then determine which points of concurrency lie on the Euler Line.

Objective: **TEXES Mathematics Competencies**

II.006.B. The beginning teacher writes equations of lines given various characteristics (e.g., two points, a point and slope, slope and y-intercept).

II.006.G. The beginning teacher models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology.

III.012.B. The beginning teacher uses properties of points, lines, angles, lengths, and distances to solve problems.

III.013.B. The beginning teacher analyzes the properties of circles and the lines that intersect them.

III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).

III.014.E. The beginning teacher applies concepts and properties of slope, midpoint, parallelism, perpendicularity, and distance to explore properties of geometric figures and solve problems in the coordinate plane.

Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.

b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

b.3.B. The student constructs and justifies statements about geometric figures and their properties.

b.4. The student uses a variety of representations to describe geometric relationships and solve problems.

c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.

d.2.B. The student uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons.

e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.

e.2.C. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of circles and the lines that intersect them.

Background: Participants must be able to write a linear equation given two points, or a point and a slope, and to solve systems of linear equations algebraically.

Materials: patty paper, centimeter ruler, compass, calculator

New Terms: centroid, circumcenter, Euler line, incenter, orthocenter

Procedures:

Part 1: Constructions with Patty Paper

Participants use patty paper to construct a perpendicular bisector, a line perpendicular to a given line from a point not on the line, and an angle bisector.

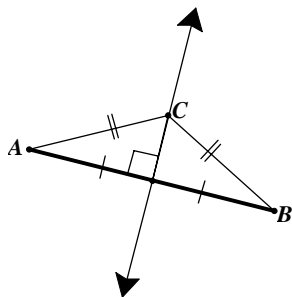
The directions for the three constructions are given on the activity sheet. Complete each construction separately. Discuss findings as each construction is completed.

The following constructions will be used in Part 2.

1. Perpendicular Bisector of a Segment

Draw a segment on a sheet of patty paper. Label the endpoints A and B . By folding, construct the perpendicular bisector of the segment. Mark a point anywhere on the perpendicular bisector. Label the point C . Draw \overline{AC} and \overline{BC} . Measure AC and BC . Write a conjecture relating C to \overline{AB} .

All points on a perpendicular bisector are equidistant from the endpoints of the bisected segment.

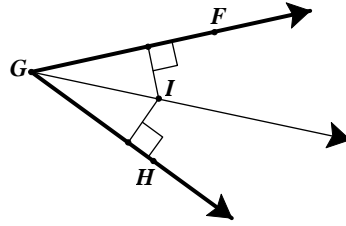


2. Perpendicular to a Segment Through a Point Not on the Segment

Draw a segment on patty paper. Mark a point, E , not on the segment. By folding, construct a perpendicular line to the segment, passing through E .

3. Angle Bisector

Draw an angle, labeled $\angle FGH$ on patty paper. By folding, construct the angle bisector. Mark a point, I , on the angle bisector. Using another sheet of patty paper for a right angle tool, or folding, construct the perpendicular lines from I to the sides of the angle, \overline{GF} and \overline{GH} . Write a conjecture relating the distance from a point on the angle bisector to the sides of the angle.



The points on an angle bisector are equidistant from the rays forming the angle.

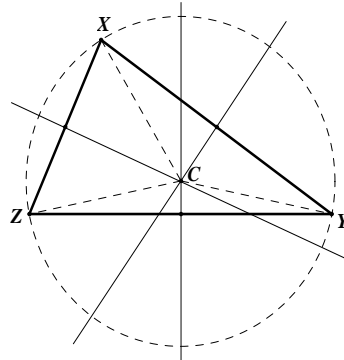
Part 2: Points of Concurrency

4. On a new sheet of patty paper, draw a scalene triangle covering at least half of the sheet. Trace the triangle onto four more sheets, so that you have five congruent triangles. Label each $\triangle XYZ$.
5. On the first triangle, construct the perpendicular bisectors of the three sides by folding. Label the point of concurrency C . Measure and compare the distances from C to the vertices of the triangle. Explain why these distances must be equal.

Since C lies on the perpendicular bisector of \overline{YZ} , from 1, $CY = CZ$.

Since C lies on the perpendicular bisector of \overline{XY} , from 1, $CX = CY$.

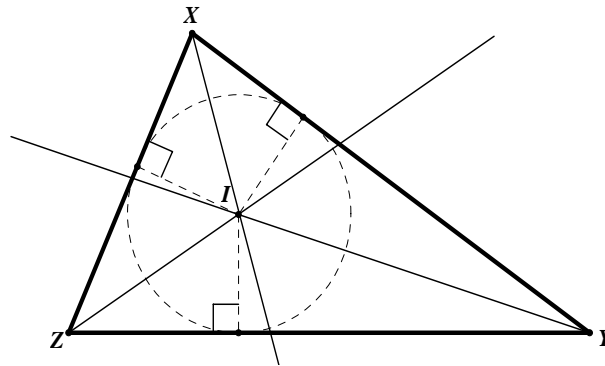
By substitution $CX = CY = CZ$.



Draw the circumscribed circle, centered at C , through the vertices of the triangle. Point C is called the circumcenter. Label the sheet “Circumcenter – Perpendicular Bisectors.”

6. On the second triangle, construct the three angle bisectors by folding. Label the point of concurrency I . Measure and compare the perpendicular distances from I to the sides of the triangle. Mark the points where the perpendiculars from I intersect the sides of the triangle. Explain why these distances must be equal.

From 2, I is equidistant from \overline{XZ} and \overline{XY} , since \overline{XI} bisects $\angle ZXY$. Similarly, I is equidistant from \overline{XZ} and \overline{YZ} . Therefore, I is equidistant from all three sides of the triangle.



Draw the inscribed circle, centered at I , through the marked points on the sides of the triangle. Point I is called the *incenter*. Label the sheet “Incenter – Angle Bisectors.”

- On the third triangle, by folding, pinch the midpoints of each of the sides of the triangle. With a ruler, draw the three medians, the segments connecting each vertex to the midpoint of the opposite sides. Label the point of concurrency M . Measure and determine the ratio into which M divides each median. Label the sheet “Centroid – Medians.”

An example is shown below.

$$ZM = 4.42 \text{ cm} \quad MT = 2.21 \text{ cm}$$

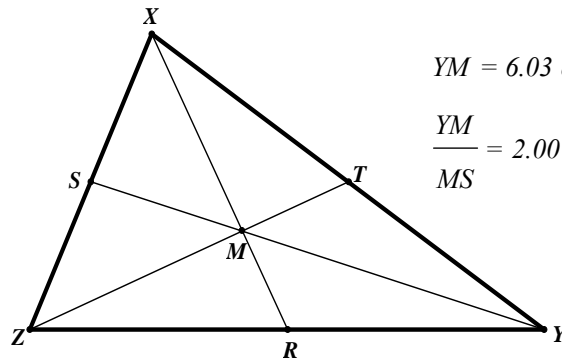
$$\frac{ZM}{MT} = 2.00$$

$$XM = 3.92 \text{ cm} \quad MR = 1.96 \text{ cm}$$

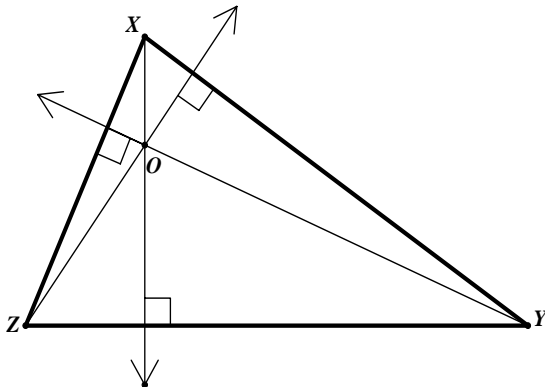
$$\frac{XM}{MR} = 2.00$$

$$YM = 6.03 \text{ cm} \quad MS = 3.02 \text{ cm}$$

$$\frac{YM}{MS} = 2.00$$

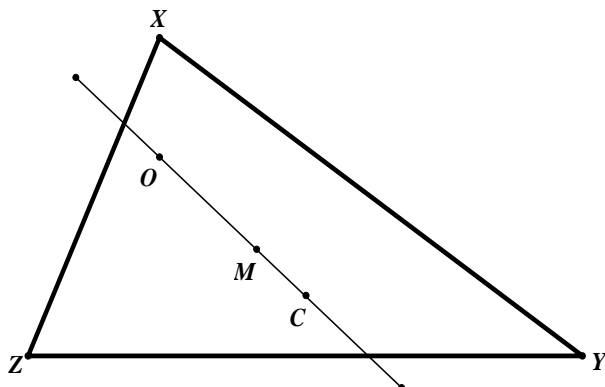


- On the fourth triangle, by folding, construct the three altitudes (the perpendicular segments from each vertex to the opposite side). If you have an obtuse triangle you will need to extend two of the sides outside the triangle. Label the point of concurrency O . Label the sheet “Orthocenter – Altitudes.”



- Place the fifth triangle directly on top of the first triangle, so that the vertices coincide exactly. Mark and label point C . Repeat with the other three points of concurrency, I ,

M , and O . Which three of the four points are collinear (lie on the same line)? Draw in the line connecting these three points. Label the sheet “Euler Line.” The Euler line is named after Leonhard Euler, a Swiss mathematician (1708-1783), who proved that the three points of concurrency are collinear.



Part 3: Algebraic Connections

The points of concurrency can be found algebraically using knowledge of slope, y -intercept, perpendicular lines, the writing of linear equations, and the solving of systems of linear equations.

In the following problems, algebraically find the circumcenter, centroid, orthocenter, and the equation of the Euler line.

Each problem requires the same algebraic work. Discuss methods for finding each point.

How do we find the circumcenter?

The *circumcenter* is the intersection of the perpendicular bisectors of the sides of the triangle. Find the equations of two of the perpendicular bisectors, and then find their intersection point. To find each equation, use the midpoint of one side of the triangle and the slope of the perpendicular to that side.

How do we find the centroid?

The *centroid* is the intersection of the medians. Find the equations of two of the medians, and then find their intersection point. To find each equation, use the midpoint of a side and the vertex opposite that side.

How do we find the orthocenter?

The *orthocenter* is the intersection of the altitudes. Find the equations of two of the altitudes, and then find their intersection point. To find each equation, use the slope of the perpendicular to a side and the vertex opposite that side.

How do we find the equation of the Euler Line?

Use two of the points of concurrency to find the equation, and then check by substituting the coordinates for the third point.

Each group should be given a different problem (10 – 12). The work for each problem may be divided among the members of the group. Then the whole group should come together to share the results and complete the problem. Participants should graph their triangles to help visualize the figures, and also to check the algebraic work. This section may be assigned for homework.

The solution for 10 follows. Answers only are given for 11 and 12.

10. Triangle ABC has vertices $A(0, 8)$, $B(-3, -1)$, and $C(5, -2)$.

$$\text{Circumcenter: } \left(\frac{3}{2}, \frac{5}{2}\right)$$

$$\text{Centroid: } \left(\frac{2}{3}, \frac{5}{3}\right)$$

$$\text{Orthocenter: } (-1, 0)$$

$$\text{Euler Line: } y = x + 1.$$

11. Triangle DEF has vertices $D(-2, 0)$, $E(-6, -2)$, and $F(3, -5)$.

$$\text{Circumcenter: } (-2, -5)$$

$$\text{Centroid: } \left(-\frac{5}{3}, -\frac{7}{3}\right)$$

$$\text{Orthocenter: } (-1, 3)$$

$$\text{Euler Line: } y = 8x + 11.$$

12. Triangle GHJ has vertices $G(3, 7)$, $H(-1, -1)$, and $J(5, -4)$.

$$\text{Circumcenter: } \left(4, \frac{3}{2}\right)$$

$$\text{Centroid: } \left(\frac{7}{3}, \frac{2}{3}\right)$$

$$\text{Orthocenter: } (-1, -1)$$

$$\text{Euler Line: } y = \frac{1}{2}x - \frac{1}{2}$$

A possible solution for 10:

$$A(0, 8), B(-3, -1), C(5, -2).$$

Circumcenter:

Midpoint of \overline{BC} :

$$\left(\frac{-3+5}{2}, \frac{-1-2}{2}\right) = \left(1, \frac{-3}{2}\right).$$

$$\text{Slope of } \overline{BC}: \frac{-1-(-2)}{-3-5} = -\frac{1}{8}.$$

Midpoint of \overline{AB} :

$$\left(\frac{0-3}{2}, \frac{8-1}{2}\right) = \left(\frac{-3}{2}, \frac{7}{2}\right).$$

$$\text{Slope of } \overline{AB}: \frac{8-(-1)}{0-(-3)} = \frac{9}{3} = 3.$$

Perpendicular slope = 8.

Equation: $y = 8x + b$.

To find b , substitute $(1, -\frac{3}{2})$ in the above equation.

$$-\frac{3}{2} = 8 + b.$$

$$b = -\frac{3}{2} - 8 = -\frac{19}{2}.$$

The equation of the perpendicular

bisector of \overline{BC} is $y = 8x - \frac{19}{2}$.

Perpendicular slope = $-\frac{1}{3}$

Equation: $y = -\frac{1}{3}x + b$.

To find b , substitute $(-\frac{3}{2}, \frac{7}{2})$ in the above equation.

$$\frac{7}{2} = -\frac{1}{3} \cdot -\frac{3}{2} + b = \frac{1}{2} + b$$

$$b = \frac{7}{2} - \frac{1}{2} = 3$$

The equation of the perpendicular

bisector of \overline{AB} is $y = -\frac{1}{3}x + 3$

Find the intersection point by substitution:

$$y = 8x - \frac{19}{2} = -\frac{1}{3}x + 3$$

$$8x + \frac{1}{3}x = 3 + \frac{19}{2}$$

$$\frac{25}{3}x = \frac{25}{2}; x = \frac{3}{2}.$$

Substitute $\frac{3}{2}$ for x in $y = -\frac{1}{3}x + 3$. $y = -\frac{1}{3} \cdot \frac{3}{2} + 3 = -\frac{1}{2} + 3 = \frac{5}{2}$.

$$x = \frac{3}{2}, y = \frac{5}{2}.$$

Check in the other equation: $y = 8x - \frac{19}{2}$. $8 \cdot \frac{3}{2} - \frac{19}{2} = \frac{24}{2} - \frac{19}{2} = \frac{5}{2}$

The circumcenter is $(\frac{3}{2}, \frac{5}{2})$.

Centroid:

Median from A to \overline{BC} :

Midpoint of \overline{BC} : $(1, -\frac{3}{2})$

A : $(0, 8)$, which is also the y -intercept.

Slope from $(0, 8)$ to $(1, -\frac{3}{2})$:

Median from C to \overline{AB} :

Midpoint of \overline{AB} : $(-\frac{3}{2}, \frac{7}{2})$

C : $(5, -2)$

Slope from $(-\frac{3}{2}, \frac{7}{2})$ to $(5, -2)$:

$$\frac{8 - (-\frac{3}{2})}{0 - 1} = \frac{\frac{19}{2}}{-1} = -\frac{19}{2}$$

The equation of the median from A to

$$\overline{BC} \text{ is } y = -\frac{19}{2}x + 8$$

$$\frac{-2 - \frac{7}{2}}{5 - (-\frac{3}{2})} = \frac{\frac{-11}{2}}{\frac{13}{2}} = -\frac{11}{13}$$

$$y = -\frac{11}{13}x + b.$$

Substitute (5, -2) to solve for b.

$$-2 = -\frac{11}{13} \cdot 5 + b$$

$$\text{Simplifying: } b = \frac{29}{13}.$$

The equation of the median from C to

$$\overline{AB} \text{ is } y = -\frac{11}{13}x + \frac{29}{13}.$$

Find the intersection point by substitution:

$$-\frac{19}{2}x + 8 = -\frac{11}{13}x + \frac{29}{13}.$$

$$-\frac{19}{2}x + \frac{11}{13}x = \frac{29}{13} - 8.$$

$$-\frac{225}{26}x = -\frac{150}{26}.$$

$$x = \frac{2}{3}. \text{ Substitute } x = \frac{2}{3} \text{ in } y = -\frac{19}{2}x + 8.$$

$$y = -\frac{19}{2} \cdot \frac{2}{3} + 8 = \frac{5}{3}.$$

$$\text{Check in the other equation: } y = -\frac{11}{13}x + \frac{29}{13} =$$

$$-\frac{11}{13} \cdot \frac{2}{3} + \frac{29}{13} = -\frac{22}{39} + \frac{87}{39} = \frac{65}{39} = \frac{5}{3}.$$

$$\text{The centroid is } (\frac{2}{3}, \frac{5}{3}).$$

Orthocenter:

$$\text{Slope of } \overline{BC}: \frac{-1 - (-2)}{-3 - 5} = \frac{1}{-8}.$$

$$\text{Perpendicular slope} = 8.$$

$$\text{Slope of } \overline{AB}: \frac{8 - (-1)}{0 - (-3)} = \frac{9}{3} = 3.$$

$$\text{Perpendicular slope} = -\frac{1}{3}$$

$A(0, 8)$ is also the y -intercept.

The equation of the altitude from A to \overline{BC} is $y = 8x + 8$.

Use $C(5, -2)$ and slope $= -\frac{1}{3}$ to find the

equation $y = -\frac{1}{3}x + b$.

Substitute $C(5, -2)$.

$$-2 = -\frac{1}{3} \cdot 5 + b.$$

$$b = -2 + \frac{5}{3} = -\frac{1}{3}.$$

The equation of the altitude from C to

\overline{AB} is $y = -\frac{1}{3}x - \frac{1}{3}$.

$$\text{Substituting: } 8x + 8 = -\frac{1}{3}x - \frac{1}{3}$$

$$8x + \frac{1}{3}x = -8 - \frac{1}{3}.$$

$$\frac{25}{3}x = -\frac{25}{3}.$$

$$x = -1.$$

Substitute $x = -1$ in $y = 8x + 8 = -8 + 8 = 0$.

$$\text{Check } y = -\frac{1}{3}x - \frac{1}{3} = 0.$$

The orthocenter is $(-1, 0)$.

Euler Line:

Use the orthocenter $(-1, 0)$ and the circumcenter $\left(\frac{3}{2}, \frac{5}{2}\right)$.

$$\text{Slope: } \frac{\frac{5}{2} - 0}{\frac{3}{2} - (-1)} = \frac{\frac{5}{2}}{\frac{5}{2}} = 1.$$

Using $(-1, 0)$, $y = 0 = 1(-1) + b = -1 + b$.

$$b = 1.$$

Euler Line: $y = x + 1$.

Ask participants to add the new terms centroid, circumcenter, Euler line, incenter, and orthocenter to their glossaries.

To close the unit, discuss the van Hiele levels represented in the activity. In 1–3 participants are performing at the Descriptive Level since participants use properties relating to reflectional symmetry. In the remainder of the activity, 4 – 12, participants apply properties at the Descriptive Level, including some Relational Level elements, such as justifications for congruence of segments relating to the circumcenter and incenter. In finding the equation of the Euler Line, participants perform at the Relational Level with respect to linear functions, since various properties of linear properties are interrelated.

The Meeting Place

Part 1: Constructions with Patty Paper

The following constructions will be used in Part 2.

1. Perpendicular Bisector of a Segment
Draw a segment on a sheet of patty paper. Label the endpoints A and B . By folding, construct the perpendicular bisector of the segment. Mark a point anywhere on the perpendicular bisector. Label the point C . Draw \overline{AC} and \overline{BC} . Measure AC and BC . Write a conjecture relating C to \overline{AB} .
2. Perpendicular to a Segment Through a Point Not on the Segment
Draw a segment on patty paper. Mark a point, E , not on the segment. By folding, construct a perpendicular line to the segment, passing through E .
3. Angle Bisector
Draw an angle, labeled $\angle FGH$ on patty paper. By folding, construct the angle bisector. Mark a point, I , on the angle bisector. Using another sheet of patty paper for a right angle tool, or by folding, construct the perpendicular lines from I to the sides of the angle, \overline{GF} and \overline{GH} . Write a conjecture relating the distance from a point on the angle bisector to the sides of the angle.

Part 2: Points of Concurrency

4. On a new sheet of patty paper, draw a scalene triangle covering at least half of the sheet. Trace the triangle onto four more sheets, so that you have five congruent triangles. Label each $\triangle XYZ$.
5. On the first triangle, construct the perpendicular bisectors of the three sides by folding. Label the point of concurrency C . Measure and compare the distances from C to the vertices of the triangle. Explain why these distances must be equal. Draw the circumscribed circle, centered at C , through the vertices of the triangle. Point C is called the circumcenter. Label the sheet "Circumcenter – Perpendicular Bisectors."
6. On the second triangle, construct the three angle bisectors by folding. Label the point of concurrency I . Measure and compare the perpendicular distances from I to the sides of the triangle. Mark the points where the

perpendiculars from I intersect the sides of the triangle. Explain why these distances must be equal. Draw the inscribed circle, centered at I , through the marked points on the sides of the triangle. Point I is called the incenter. Label the sheet “Incenter – Angle Bisectors.”

7. On the third triangle, by folding, pinch the midpoints of each of the sides of the triangle. With a ruler, draw the three medians, the segments connecting each vertex to the midpoint of the opposite sides. Label the point of concurrency M . Measure and determine the ratio into which M divides each median. Label the sheet “Centroid – Medians.”
8. On the fourth triangle, by folding, construct the three altitudes (the perpendicular segments from each vertex to the opposite side). If you have an obtuse triangle you will need to extend two of the sides outside the triangle. Label the point of concurrency O . Label the sheet “Orthocenter – Altitudes.”
9. Place the fifth triangle directly on top of the first triangle, so that the vertices coincide exactly. Mark and label point C . Repeat with the other three points of concurrency, I , M , and O . Which three of the four points are collinear (lie on the same line)? Draw in the line connecting these three points. Label the sheet “Euler Line.” The Euler line is named after Leonhard Euler, a Swiss mathematician (1708-1783), who proved that three points of concurrency are collinear.

Part 3: Algebraic Connections

The points of concurrency can be found algebraically using knowledge of slope, y -intercept, perpendicular lines, the writing of linear equations, and the solving of systems of linear equations.

In the following problems, algebraically find the circumcenter, centroid, orthocenter, and the equation of the Euler line.

10. Triangle ABC has vertices $A (0, 8)$, $B (-3, -1)$, and $C (5, -2)$.
11. Triangle DEF has vertices $D (-2, 0)$, $E (-6, -2)$, and $F (3, -5)$.
12. Triangle GHJ has vertices $G (3, 7)$, $H (-1, -1)$, and $J (5, -4)$.

References and Additional Resources

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