Connecting with Conics

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Connecting Conics with Geometry

Two constructions, performed with patty paper and/or the TI-Nspire can provide deeper understanding of the conic sections.

Construction #1

Using a compass, draw Circle *P* on a piece of patty paper. Label another point A other than the center in the interior of the circle. Select a point on the circle and fold the patty paper so that the point lies on top of A. Create the crease. Repeat this process for other points on the circle. Which figure is created? Why?

Now re-create this construction using the TI-Nspire.

Construction #2

Next, using a compass, draw Circle Q on another piece of patty paper. Label another point B on the exterior of the circle. Take a point on the circle and fold the patty paper so that the point lies on top of B. Create the crease. Repeat the process for other points on the circle. What figure is created? Why?

Now re-create this construction using the TI-Nspire.

Extension

How can paper folding be used to model the parabola?

Connection Conics with Algebra and Trigonometry – Parametric Representation

The Unit Circle

The unit circle is easily represented by the parametric equations:

$$\begin{cases} x_1(t) = \cos(t) \\ y_1(t) = \sin(t) \end{cases}$$

Ellipses

A comparison of the Pythagorean identity: $\cos^2 t + \sin^2 t = 1$

and a standard form for the equation of an ellipse : $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

allows for two simple substitutions : $\cos^2 t = \dfrac{(x-h)^2}{a^2}$ and $\sin^2 t = \dfrac{(y-k)^2}{b^2}$

Solving these two equations for x and y yields a pair of parametric equations:

$$x = a\cos t + h$$

$$y = b\sin t + k$$

Re-express $\dfrac{(x-3)^2}{9} + \dfrac{(y+2)^2}{4} = 1$ using parametric equations and graph. Use degree mode and graph in an appropriate square window for $0 \le t \le 360°$.

A few personal comments are important at this point:

I chose the substitutions that I did to reinforce the use of *x* and *y* coordinates of a unit circle to represent cosine and sine respectively. In using this method I am de-emphasizing the idea that "*a*" corresponds to the major axis, etc. I focus on the idea that "*a*" is a stretch in the *x* equation and therefore a horizontal stretch. Likewise, "*b*" is a vertical stretch. I'd just as soon not use the letters "*a*" and "*b*" at all, but rather, focus on the major axis being the axis with the "largest" stretch.

Some students see a contradiction in the transformation in parametric representation when compared to the Cartesian representation. By re-writing $x = 3\cos t + 3$ in the form $x - 3 = 3\cos t$, I try to show that there is really no contradiction.

Hyperbolas

By using the Pythagorean identity: $\sec^2 t - \tan^2 t = 1$

and a standard form for a hyperbola : $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$

One can derive the following pairs of parametric equations to represent hyperbolas:

$x = a\sec t + h$	$x = b\tan t + h$
$y = b\tan t + k$	$y = a\sec t + k$
(horizontal transverse axis)	(vertical transverse axis)

In a hyperbola, unlike an ellipse, it makes a difference which trigonometric function corresponds with which variable.

> Using the same window settings as before, re-express the equation parametrically to graph the hyperbola $\dfrac{(y+2)^2}{4} - \dfrac{(x-1)^2}{9} = 1$.

Parabolas

Parabolas are most easily graphed parametrically without the use of trigonometric functions. All non-rotated parabolas can either be written in the form $y = f(x)$ or $x = f(y)$. Parametrically, parabolas that can be written $y = f(x)$ can be graphed using $x = t$ and $y = f(t)$, likewise parabolas that can be represented as $x = f(y)$ can be graphed parametrically using $x = f(t)$ and $y = t$. In this case the *t*-step of the window must be adjusted to include negative values for *t* or the entire parabola will not appear.

Extensions

Exploring rotated conic sections is an extension of this work. To do this view a pair of parametric equations as a 2 x 1 vector matrix; $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$,

then right-multiply this matrix by a rotation matrix; $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

The resultant 2 x1 matrix; $\begin{bmatrix} \cos\theta \cdot x(t) - \sin\theta \cdot y(t) \\ \sin\theta \cdot x(t) + \cos\theta \cdot y(t) \end{bmatrix}$

represents a new pair of parametric equations that rotate the conic θ degrees counter-clockwise.

In the same window graph the hyperbola from the previous example and the same hyperbola rotated 45°