Graphs are Everywhere!

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Computational and Applied Mathematics
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Summer Math Days
Rice University
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Me and Mathematics

Senior Student Council members Harold Jackson and Alan Shipp discuss plans for a project after a Student Council meeting.

Senior Illya Hicks, a member of the offensive line and an Academic All-American, sits back and relaxes during calculus.
My Story
I Love Texas
I also still love football!
I. Basic Definitions

II. Different Graph Applications

III. Dominating Sets, TSP, Clique & $k$-plexes

IV. Conclusions
Graphs (Networks)

Graph $G=(V, E)$

- Vertex set $V$ is finite
- Edges $E = \{uv : u, v \in V\}$
- Undirected (for this talk)
- $u$ is a neighbor of $w$ if $uw \in E$

clique

neighbors of $v$

stable set
I Can Tell You My Secret Now?

I see graphs everywhere!
Network (Graph)
Applications

- vertices represent actors: people, places, companies
- edges represent ties or relationships
- Applications
  - Criminal network analysis
  - Data mining
  - Wireless Networks
  - Genes Therapy
  - Biological Neural Networks
Van Gogh Graph

Provided by Don Johnson, Rice
Gene Co-expression Networks

vertices represent genes
edges represent high correlation between genes
(Carlson et al. 2006)
Biological Neural Networks

vertices represent neurons
(Berry and Temman 2005)
Social Network Pop Quiz
9-11 Terrorist Network

1) Alshehri
2) Sugami
3) Al-Marabh
4) Hijazi
5) W. Alshehri
6) A. Alghamdi
7) M. Alshehri
8) S. Alghamdi
9) Ahmed
10) Al-Hisawi
11) Al-Omari
12) H. Alghamdi
13) Alnami
14) Al-Haznawi
15) Darkazanli
16) Abdi
17) Al-Shehhi
18) Essabar
19) S. Alhazmi
20) N. Alhazmi
21) Bahaji
22) Jarrah
23) Atta
24) Shaikh
25) El Motassadeq
26) Al-Mihdhar
27) Moussaoui
28) Al-Shibh
29) Raissi
30) Hanjour
31) Awadallah
32) Budiman
33) Al-ani
34) Moqed
35) Abdullah
36) Al Salmi
37) Alhazmi
Do You Like Bacon?
Dominating Set
Dominating Set
Minimum Dominating Set

- A **dominating set** $D$ is a subset of vertices in a graph $G$ such that every vertex of $G$ is either a member of $D$ or is adjacent to a member of $D$.

- Applications
  - Sensor Networks
  - Marketing
  - Ad-hoc mobile networks (robots, cell phones)
  - Ship warehouse design
Health Logistics

Amber Kunkel, Elizabeth Van Itallie, Duo Wu
Mission Impossible: Rogue Nation

IMF instructions to Ethan Hunt:

- Starting from home base, visit cities \( \{c_2, c_3, \ldots, c_n\} \) to do covert operations and come back to home base.
- You can not visit any city twice!
- Since the agency is under budget cuts, you must complete your mission with lowest possible travel distance.
An Example
Complexity of the Mission

- In general, there are \((n-1)!/2\) possible solutions.

- Suppose you could evaluate a possible solution in one nanosecond \((10^{-9} \text{ seconds})\). If the number of cities were 23, then it would take you 178 centuries to look at all possible solutions.
The Traveling Salesman Problem

Given a finite number of “cities” along with the cost of travel between each pair of them. Find the cheapest way to visit all the “cities” and return to your starting point.

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<th>Cities</th>
<th>Who?</th>
<th>Year</th>
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</table>
World TSP


Keld Helsgaun’s Tour: 7,515,790,354 LP Bound: 7,512,218,268 Gap: 0.0476%
Mona Lisa

$1,000 for shorter tour

100,000 Cities, Robert Bosch, February 2009

Yuichi Nagata’s Tour: 5,757,191  LP Bound: 5,757,046  Gap: 0.0025%
A graph is a **clique** if every vertex is adjacent to the rest of vertices.
Cliques
Maximum Clique

• A **clique** is a subset of nodes such that there is an edge between any two nodes in the set.

• two nodes can’t be in a clique together if they are not adjacent

• Applications
  • Bioinformatics
  • Social networks
  • Online auctions
Homer Ignoring Lisa
Homer ignoring Lisa (en español)
The Simpsons Social Network

[Diagram showing characters from The Simpsons in a network structure]
What is cohesiveness in terms of graphs?

- Debated by social scientists
- Three general properties
  - Familiarity (few strangers)
  - Members can easily reach each other (quick communication)
  - Robustness (not easily destroyed by removing members)
Is this graph cohesive?

Clique is too restrictive!
Different versions of cohesiveness

- Relax distance requirement between members
  - $k$-clique (Luce 1950)
  - $k$-club (Alba 1973)

- Relax the familiarity (# of neighbors) between members
  - $k$-plex (Siedman & Foster 1978)
  - $k$-core (Siedman 1983)
**k-plexes**

- Given a graph $G=(V, E)$ and some integer $k > 0$, a set $S \subseteq V$ is called a **$k$-plex** if every node of $S$ has at most $k-1$ non-neighbors in $S$

- Cliques are 1-plexes

- NP-hard to find maximum $k$-plex, $\omega_k(G)$, in a graph $G$
1-plexes are cliques
2-plexes

at most 1 non-neighbor
9-11 Terrorist Network

1) Alshehri
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36) Al Salmi
37) Alhazmi
Ready for Co-k-plexes!!!
Another Example: Retail Location

Stable set
Starbucks in Springfield
Another Example: Retail Location
$k$-plexes and co-$k$-plexes
My Research: Combinatorial Optimization

• How can we find the largest k-plex in a graph?

• Two ways I attack problems
  • Combinatorial (graph) algorithms
  • Polyhedral Combinatorics
Graph Coloring

$\omega(G) \leq \chi(G)$
Co-$k$-plex Coloring

$\omega_k(G) \leq \chi_k(G)$
Linear and Integer Programming

max $3x_1 + 2x_2$

$-x_1 + 2x_2 \leq 4$

$5x_1 + 1x_2 \leq 20$

$-2x_1 - 2x_2 \leq -7$

$x_1, x_2 \geq 0$

$x$ integer

facet
Wrap-Up

- Graph Definitions
- Applications
- Dominating Sets, TSP, Cliques & k-plexes
Polyhedral Approach

• Let $N[v]$ denote the closed neighborhood of vertex $v$
• Let $d(v)$ denote $|V \setminus N[v]|$

\[
\begin{align*}
\text{Max} \sum_{v \in V} x_v \\
\text{st.} \sum_{u \in V \setminus N[v]} x_u & \leq (k - 1)x_v + d(v)(1 - x_v) \quad \forall v \in V \\
x_v & \in \{0, 1\} \quad \forall v \in V
\end{align*}
\]
Polyhedral Approach
Acknowledgments

• My collaborator: Ben McClosky, Ph. D.

• NSF
  • DMI 0521209
  • DMS 0611723
  • CMMI 0926618
Any Questions?
Relevant Literature

- Seidman & Foster (1978)
  - Introduced $k$-plexes in context of social network analysis

- Balasundaram, Butenko, Hicks, and Sachdeva (2006)
  - IP formulation for maximum $k$-plex problem
  - NP-complete complexity result

- McClosky & Hicks (2007)
  - Co-2-plex polytope

- McClosky & Hicks (2008)
  - Graph algorithm to compute $k$-plexes
Co-\(k\)-plexes

- Given a graph \(G = (V, E)\), a set \(S \subseteq V\) is called a co-\(k\)-plex if \(\Delta(G[S]) \leq k - 1\), where \(\Delta\) denotes maximum degree.

- Stable sets are co-1-plexes and co-\(k\)-plexes form independence systems.

- NP-hard to find maximum co-\(k\)-plex, \(\alpha_k(G)\) in a graph \(G\).

- Co-2-plexes correspond to vertex induced subgraphs of isolated nodes and matched pairs.
Co-k-plex Polytope

- Given graph $G$, let $\mathcal{J}^k$ be the set of co-k-plexes in $G$
- For all $S \in \mathcal{J}^k$, let $x^S$ be the incidence vector for $S$.
- Define $P_k(G) = \text{conv}(\{x^S : S \in \mathcal{J}^k\})$
- $P_2(G)$ shares many properties with $P_1(G)$
Co-2-plex analogs

- Padberg (1973)
  - Clique and odd hole inequalities
- Trotter (1975)
  - Web inequalities
- Minty (1980)
  - claw-free graphs
2-plex Inequalities

- Theorem (Padberg): If $K$ is a maximal clique in $G$, then $\sum_{v \in K} x_v \leq 1$ is a facet for $P_1(G)$.

- Theorem (M & H, B et al.): If $K$ is a maximal 2-plex in $G$ such that $|K| > 2$, then $\sum_{v \in K} x_v \leq 2$ is a facet for $P_2(G)$.
Odd-mod Hole Inequalities

- **Theorem (Padberg):** If $C$ is an $n$-chordless cycle such that $n > 3$ is odd, then $\sum_{v \in V(C)} x_v \leq \lfloor n/2 \rfloor$ is a facet for $P_1(C)$.

- **Theorem (M & H):** If $C$ is an $n$-chordless cycle such that $n > 2$ and $n \neq 0 \mod 3$, then $\sum_{v \in V(C)} x_v \leq \lfloor 2n/3 \rfloor$ is a facet for $P_2(C)$.
Webs

- For fixed integers $n \geq 1$ and $p$ such that $1 \leq p \leq \lfloor n/2 \rfloor$, the web $W(n, p)$ has $n$ vertices and edges $E = \{(i, j): j = i + p, \ldots, i + n - p; \forall \text{ vertices } i\}$
Theorem (Trotter): If \( \gcd(n, p) = 1 \), then 
\[
\sum_{v \in V(W(n,p))} x_v \leq p
\]
is a facet for \( P_1(W(n, p)) \).

Theorem (M & H): If \( \gcd(n, p + 1) = 1 \), then 
\[
\sum_{v \in V(W(n,p))} x_v \leq p + 1
\]
is a facet for \( P_2(W(n, p)) \).
Given an integer $k \geq 1$, the graph $G$ is a $k$-claw if there exists a vertex $v$ of $G$ such that $V(G) = \mathcal{N}[v]$, $\mathcal{N}(v)$ is a co-$k$-plex, and $|\mathcal{N}(v)| \geq \max\{3, k\}$.
2-claw free graphs

- **Theorem (B & H):** A graph $G$ is 2-claw free if and only if $\Delta(G) \leq 2$ or $G$ is 2-plex.

- This theorem will be used to describe a class of 0-1 matrices $A$ for which the polytope $P=\{x \in \mathbb{R}^n_+: Ax \leq 2, x \leq 1\}$ is integral.
A clutter is a pair \((V, E)\) where \(V\) is a finite set and \(E\) is a family of subsets of \(V\) none of which is included in another.
Clutters of Maximal 2-plexes

- Given a graph $G$, let $C$ be the clutter whose vertices are $V(G)$ and whose edges are maximal 2-plexes of $G$. 

![Graph G](image1)

![Clutter C](image2)
Let A be the edge-vertex incidence matrix of C.

- **Theorem (M & H):** Let A be the 2-plex clutter matrix of G. The polytope $P = \{ x \in \mathbb{R}^n_+: Ax \leq 2, x \leq 1 \}$ is integral if and only if the components of G are 2-plexes, co-2-plexes, paths, or 0 mod 3 chordless cycles.

- **Corollary (M & H):** Given a 2-plex clutter matrix A, there is a polynomial-time algorithm to determine if $P = \{ x \in \mathbb{R}^n_+: Ax \leq 2, x \leq 1 \}$ is integral.
Future Work

- Combinatorial algorithm to compute maximum k-plexes (involves k-plex coloring)
- Find facets of $P_k(G)$ for $k > 2$.
- Can k-plex clutter matrices give insight in polyhedra defined as $P=\{x \in \mathbb{R}^n_+: Ax \leq k, x \leq 1\}$?
Other inequalities

• **Stable Sets**
  - \( \sum_{v \in I} x_v \leq k \ \forall \text{stable sets } I \ \text{s.t. } |I| \geq k+1 \)

• **Holes**
  - \( \sum_{v \in H} x_v \leq k + 1 \ \forall \text{holes } H \ \text{s.t. } |H| \geq k+3 \)

• **Co-k-plexes**
  - \( \sum_{v \in S} x_v \leq \omega_k(S) \ \forall \text{co-k-plexes } S \)
2-plex Computational Results

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<th>G</th>
<th>n</th>
<th>m</th>
<th>density</th>
<th>(\omega(G))</th>
<th>BIS</th>
<th>UB</th>
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Who was the first African-American to receive a PhD in Mathematics?
Elbert F. Cox

Dissertation: Polynomial Solutions of Difference Equations
Ph.D. Cornell University, 1925
Advisor: William Lloyd Garrison
Who was the first African-American to receive a PhD in Mathematics at Rice University?
Raymond Johnson

Dissertation: A Priori Estimates and Unique Continuation Theorems for Second Order Parabolic Equations
Ph.D. Rice University, 1969
Advisor: Jim Douglass Jr.