TRIGONOMETRIC PROOFS OF CONGRUENT TRIANGLES

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**Types of proofs**

**Paragraph Proof** – This proof consists of a detailed paragraph explaining the proof process. The paragraph contains steps and supporting justifications which prove the statement true.

**Two column proof** – This proof consists of two columns, where the first column contains a numbered chronological list of steps, called *Statements*, leading to the desired conclusion. The second column contains the justifications, called *Reasons*, to support each step in the proof.

**Flow Proof (Chart Proof)** – This proof format shows the structure of a proof using boxes and connecting arrows. The appearance is like a detailed drawing of the proof or perhaps a graphical organizer.
Trends in Proofs

- Traditional Deductive Proofs
- Demonstrating a Theorem (numerical/algebraic calculations or measurements)
- Hands-On Activities (with some materials and manipulatives)
- Audio/Visual Materials (traditional lecturing or cartoons)
## Statements about Congruent Triangles

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### Postulate 4-1 Side-Side-Side (SSS) Postulate

**Postulate**
If the three sides of one triangle are congruent to the three sides of another triangle, then the two triangles are congruent.

If . . .
\[ AB \cong DE, \quad BC \cong EF, \quad AC \cong DF \]

Then . . .
\[ \triangle ABC \cong \triangle DEF \]

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### Postulate 4-2 Side-Angle-Side (SAS) Postulate

**Postulate**
If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the two triangles are congruent.

If . . .
\[ AB \cong DE, \quad \angle A \cong \angle D, \quad AC \cong DF \]

Then . . .
\[ \triangle ABC \cong \triangle DEF \]

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### Postulate 4-3 Angle-Side-Angle (ASA) Postulate

**Postulate**
If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the two triangles are congruent.

If . . .
\[ \angle A \cong \angle D, \quad AC \cong DF, \quad \angle C \cong \angle F \]

Then . . .
\[ \triangle ABC \cong \triangle DEF \]
**Theorem 4-6 Hypotenuse-Leg (HL) Theorem**

If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and a leg of another right triangle, then the triangles are congruent.

If \( \triangle PQR \) and \( \triangle XYZ \) are right \( \triangle \), \( PR = XZ \), and \( PQ = XY \)

Then \( \triangle PQR = \triangle XYZ \)

For a proof of Theorem 4-6, see the Reference section on page 683.

**Proof**

Given: \( \triangle PQR \) and \( \triangle XYZ \) are right triangles, with right angles \( Q \) and \( Y \). \( PR = XZ \) and \( PQ = XY \).

Prove: \( \triangle PQR = \triangle XYZ \)

Proof: On \( \triangle XYZ \), draw \( 

Mark point \( S \) so that \( YS = QR \). Then, \( \triangle PQR \equiv \triangle XYS \) by SAS.

Since corresponding parts of congruent triangles are congruent, \( PR = XS \). It is given that \( PR = XZ \), so \( XS = XZ \) by the Transitive Property of Congruence. By the Isosceles Triangle Theorem, \( \triangle S \equiv \triangle Z \), so \( \triangle XYS \equiv \triangle XYZ \) by AAS. Therefore, \( \triangle PQR \equiv \triangle XYZ \) by the Transitive Property of Congruence.
PROOF AS A SEQUENCE OF ALGEBRAIC/TRIGONOMETRIC CALCULATIONS

Prerequisites:

• Cosine Theorem (Law of Cosines)
• Sine Theorem (Law of Sines)
• Interior Angles Theorem
A triangle consists of 3 sides and 3 angles

$$< s_1, s_2, s_3, s_4, s_5, s_6 > = < a, b, c, \angle A, \angle B, \angle C >$$

Algebraically, a triangle is term of six numbers (elements):

$$< s_1, s_2, s_3, s_4, s_5, s_6 >$$

**DEFINITION:**
If for any given three elements, it is possible to find some unique formulae for the other three elements, then all triangles with the same given data are congruent.

$$s_{i_1} = f(s_{i_4}, s_{i_5}, s_{i_6})$$
$$s_{i_2} = \phi(s_{i_4}, s_{i_5}, s_{i_6})$$
$$s_{i_3} = \psi(s_{i_4}, s_{i_5}, s_{i_6})$$
Given: sides $b$, $c$ and the angle $\angle A$
Find: side $a$, angle $\angle B$, angle $\angle C$

Find:

$$a = f(b, c, \angle A)$$
$$\angle B = \phi(b, c, \angle A)$$
$$\angle C = \psi(b, c, \angle A)$$
SAS THEOREM (CONT’D)

Process:

\[ a^2 = b^2 + c^2 - 2bc \cdot \cos(A) \]
\[ \frac{a}{\sin(A)} = \frac{b}{\sin(B)} \]
\[ m\angle A + m\angle B + m\angle C = \pi \]

Solution:

If \( b \cdot \cos A > c \), then \( \angle B \) is acute and \( m\angle B = \arcsin\left(\frac{b}{c} \cdot \sin A\right) \)

If \( b \cdot \cos A < c \), then \( \angle B \) is obtuse and \( m\angle B = \pi - \arcsin\left(\frac{b}{c} \cdot \sin A\right) \)

If \( b \cdot \cos A = c \), then \( \angle B \) is a right angle and \( m\angle B = 90^\circ \)
Final answer

\[ a = \sqrt{b^2 + c^2 - 2bc \cdot \cos A} \]

\[ m_\angle B = \arcsin\left(\frac{b}{c} \cdot \sin A\right) \quad \text{if } b \cdot \cos A > c \]

\[ m_\angle B = \pi - \arcsin\left(\frac{b}{c} \cdot \sin A\right) \quad \text{if } b \cdot \cos A < c \]

\[ m_\angle B = \frac{\pi}{2} \quad \text{if } b \cdot \cos A = c \]

\[ m_\angle C = \pi - m_\angle A - m_\angle B \]

END OF THEOREM
Given: side $c$, $m\angle A$, $m\angle B$
Find: side $b$, side $a$, $m\angle C$

\[ a = f(c, m\angle A, m\angle B) \]
\[ b = f(c, m\angle A, m\angle B) \]
\[ m\angle C = \psi(c, m\angle A, m\angle B) \]
Process:

\[ a^2 = b^2 + c^2 - 2bc \cdot \cos(A) \]

\[ \frac{a}{\sin(A)} = \frac{b}{\sin(B)} \]

\[ m\angle A + m\angle B + m\angle C = \pi \]

Solution

\[ m\angle C = \pi - m\angle A - m\angle B \]

\[ a = \frac{c \cdot \sin(A)}{\sin(\pi - m\angle A - m\angle B)} \]

\[ b = \frac{c \cdot \sin(B)}{\sin(\pi - m\angle A - m\angle B)} \]

End of Theorem
Given: side $a$, side $b$, and side $c$
Find: angle $\angle A$, angle $\angle B$, and $\angle C$

$$m\angle A = f(a, b, c)$$
$$m\angle B = f(a, b, c)$$
$$m\angle C = f(a, b, c)$$
SSS THEOREM

Process

Solution

End of Theorem
SSA - THE CASE OF AMBIGUITY
SSA Theorem

Given: side a, side b, angle $\angle A$ (is not between sides a and b)
Find: $\angle B$, $\angle C$, side C

No solution if $b \cdot \sin A > a$
Two solutions if $b \cdot \sin A < a < b$
One solution if $b \cdot \sin A = a$ or $b < a$
SSA Theorem (con’d)

Process:

\[
\frac{a}{\sin A} = \frac{b}{\sin B}
\]

\[m_\angle A + m_\angle B + m_\angle C = \pi\]

\[c = \sqrt{a^2 + b^2 - 2ab \cdot \cos C}\]

Solution:

\[m_\angle B = \arcsin \left(\frac{b}{a} \cdot \sin A\right), \text{ if } \angle B \text{ is acute}\]

\[m_\angle B = \pi - \arcsin \left(\frac{b}{a} \cdot \sin A\right), \text{ if } \angle B \text{ is obtuse}\]

\[m_\angle C = \pi - m_\angle A - m_\angle B\]

\[c = \sqrt{a^2 + b^2 - 2ab \cdot \cos C}\]

END OF THEOREM
Statements about congruent triangles (SAS, ASA, SSS, HL) are THEOREMS, not postulates or conjectures.

Proofs of congruent triangles can be implemented into Trigonometry courses.

Other geometric theorems could be proved as well with algebraic and trigonometric methods.
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