RUSMP/MLI Colloquium

Tropical Mathematics

An Interesting and Useful Variant of Ordinary Arithmetic

June 8, 2005
Tropical Mathematics

A new mathematics

- Starts with a new arithmetic
- Includes polynomials, curves, higher algebra
- Useful in combinatorics, algebraic geometry
- Useful in genetics
- It is fun to do math in a different setting
Why Tropical Mathematics?

- Coined by French mathematicians
- In honor of Imre Simon, a Brazilian mathematician
- The name simply reflects how a few Frenchmen view Brazil
Tropical Arithmetic

- Ordinary arithmetic
  - Real numbers, addition ($+$) and multiplication ($\times$)

- Tropical arithmetic
  - Real numbers plus infinity, denoted by $\infty$
  - Tropical addition ($\oplus$)
  - Tropical multiplication ($\otimes$)
Tropical Addition

\( a \oplus b = \text{the minimum of } a \text{ and } b. \)

- **Examples:**

  \[
  3 \oplus 5 = 3, \quad 3 \oplus (-5) = -5 \\
  12 \oplus 0 = 0, \quad 0 \oplus (-3) = -3 
  \]

- **The additive unit is \( \infty \).**

  \[
  \infty \oplus 3 = 3 \\
  \infty \oplus x = x \oplus \infty = x \text{ for all } x
  \]
# Tropical Addition Table

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Differences

- Subtraction is not always possible.
  - The equation $3 \oplus x = 5$ has no solution.
  - The equation $3 \oplus x = 1$ has a solution.
  - The equation $a \oplus x = \infty$ has no solution if $a \neq \infty$.
- We have to stay away from looking for solutions to equations.
Tropical Multiplication

- \( a \otimes b = a + b \)
  
  Tropical multiplication is the same as ordinary addition.

- Examples:
  
  \[ 3 \otimes 5 = 8, \quad 3 \otimes (-5) = -2, \]
  
  \[ (-1) \otimes 3 = 2, \quad 1 \otimes 13 = 14. \]

- The multiplicative unit is 0.
  
  \[ 0 \otimes 13 = 13. \]
  
  \[ 0 \otimes x = x \otimes 0 = x \text{ for all } x. \]
# Tropical Multiplication Table

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Similarities and Differences

- Commutative laws are valid
- The distributive law still holds.
- \((x \oplus y)^3 = x^3 \oplus y^3\)
Linear Functions

\[ y = 5 \]

\[ y = 3 \otimes x \]

\[ y = 3 \otimes x \oplus 5 \]
Linear Functions

- The graph of $y = 5$ is a straight line with slope 0.
- The graph of $y = 3 \otimes x$ is a straight line with slope 1.
- The graph of $y = 3 \otimes x \oplus 5$ is a crooked line.
- Notice:

  \[
  3 \otimes x \oplus 5 = \min\{x + 3, 5\}
  \]
  \[
  = 3 + \min\{x, 2\}
  \]
  \[
  = 3 \otimes (x \oplus 2)
  \]

  $x = 2$ is where the graph bends.
Monomials

• Monomials:

\[ x^2 = x \otimes x = x + x = 2x \]
\[ x^3 = x \otimes x \otimes x = 3x \]
\[ x^p = p \times x \]

♦ Monomials are linear functions with integer coefficients.

• \( 3 \otimes x^2 = 3 + (2x) \)

♦ The graph is a line with slope 2.

• \( 4 \otimes x^3 = 3x + 4 \)

♦ The graph is a line with slope 3.

• The exponent is the slope of the graph.
Polynomials

Example 1:

\[ p(x) = 2 \otimes x^2 \oplus x \oplus 5 \]

\[ = \min\{2x + 2, x, 5\} \]

- The graph is a twice bent line.
  - The graph bends at \( x = -2 \) and \( x = 5 \).
- We can show that \( p(x) = 2 \otimes [x \oplus (-2)] \otimes [x \oplus 5] \)
Example 2:

\[ p(x) = x^2 \oplus 3 \otimes x \oplus 2 \]

\[ = \min\{2x, x + 3, 2\} \]

- The graph is a once bent line.
  - The graph bends at \( x = 1 \)
- We can show that \( p(x) = (x \oplus 1)^2 \)
Factorization of Polynomials

- Our two example polynomials factor into linear factors.
  - The factors have the form $x \oplus a$, where $a$ is a bend point for the graph.
- Any tropical polynomial can be expressed in one and only one way as the product of linear factors.
  - Thus the Fundamental Theorem of Algebra remains true in tropical mathematics.
  - The factors are of the form $x \oplus a$, where $a$ is a bend point for the graphs of the function. All such factors occur.
Polynomials in Two Variables

• A monomial represents a linear function.
  ♦ Example: \( p(x, y) = 3 \otimes x \otimes y = 3 + x + y \)

• A polynomial represents the minimum of one or more linear functions.
  ♦ Example: \( p(x, y) = x \oplus y \oplus 1 = \min\{x, y, 1\} \)

• The bend points of the graph occur where two or more of the linear functions agree.
Curves

• In ordinary math, the zero set of $x^2 + y^2 - 1$ is a circle — a curve.

• In tropical math, the zero set is replaced with the bend set — a tropical curve.

• Examples
  
  ♦ 1. $p(x, y) = x \oplus y \oplus 1 = \min\{x, y, 1\}$
  
  ♦ 2. $p(x, y) = x^2 \oplus y^2 \oplus 4 = \min\{2x, 2y, 4\}$
  
  ♦ 3. $p(x, y) = x^2 \oplus y^2 \oplus x \oplus 4 = \min\{2x, 2y, x, 4\}$
The End
$y = 2 \otimes x^2$

$p(x) = 2 \otimes x^2 \oplus x \oplus 5$

$y = x$

$y = 5$
\[
\begin{align*}
\text{y} &= \text{x}^2 \\
\text{y} &= 2 \\
p(x) &= \text{x}^2 \oplus 3 \otimes \text{x} \oplus 2 \\
y &= 3 \otimes \text{x}
\end{align*}
\]
p(x,y) = x \oplus y \oplus 1
\[ p(x,y) = x^2 \oplus y^2 \oplus 4 \]

\[ x^2 \ (= 2x) \]

\[ y^2 \ (= 2y) \]
\[ p(x, y) = x^2 \oplus y^2 \oplus x \oplus 4 \]

- \( x^2 \) (\(= 2x\))
- \( y^2 \) (\(= 2y\))