## Addition of Fractions

 The Unrecognized ProblemA Preesentation by

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at the
Conference for the Advancement of


## A Little Pretest

## Question 1:



## A Little Pretest

Question 2:
A family travels from Houston to San Antonio.
Because of traffic, they only average 40 mph going there.
On the return trip the traffic is much better, so they average 50 mph .
What is their average speed for the entire trip?

## A Little Pretest

## Question 3:



## Answer:

## Question 1:

$$
\frac{1}{2}+\frac{1}{3}=\frac{2}{5}
$$



$\qquad$
$\frac{2}{5}$


$$
\begin{aligned}
& \frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6} \\
& \frac{a}{b}+\frac{c}{d}=\frac{a d+b c}{b d}
\end{aligned}
$$

We add the parts, but the whole is the same.


$$
\begin{aligned}
& \frac{1}{4}+\frac{1}{4}=\frac{2}{8} \\
& \frac{a}{b} \oplus \frac{c}{d}=\frac{a+c}{b+d}
\end{aligned}
$$

Parts and wholes are both added.
Parts are the same, but the whole is larger.


## "Crazy"Addition

Makes sense when sets of things are combined. Operation must be done with original data.

$$
\frac{4}{6} \oplus \frac{1}{2}=\frac{5}{8}
$$

Key concept is accumulation.

## Farey Fractions

John Farey, Sr. (1766-1826) British geologist Sequence of reduced fractions Called this "sum" the mediant Rational Mean

$$
\frac{1}{4} \frac{2}{7} \frac{1}{3} \frac{2}{5} \frac{1}{2}
$$

## Jose Altuve

0.303 hits/at bat
0.299 hits/at bat

$\frac{96 \text { hits }}{317 \text { at bats }} \oplus \frac{0 \text { hits }}{4 \text { at bats }}=\frac{96 \text { hits }}{321 \text { at bats }}$

## Mixtures

5 liters of a $40 \%$ acid solution.
20 liters of a $50 \%$ acid solution.
What is the concentration of the new solution?

$$
\frac{2 \mathrm{~L} \text { of acid }}{5 \mathrm{~L} \text { of solution }} \oplus \frac{10 \mathrm{~L} \text { of acid }}{20 \mathrm{~L} \text { of solution }}=\frac{12 \mathrm{~L} \text { of acid }}{25 \mathrm{~L} \text { of solution }}
$$

$48 \%$ solution

## Answer:

## Question 2:

A family travels from Houston to San Antonio.
They average 40 mph going there.
They average 50 mph coming back.
What is their average speed for the entire trip?
Assume the distance each way to be 200 miles.
$200 \mathrm{mi} @ 40 \mathrm{mph}$ is $\frac{200 \mathrm{mi}}{5 \mathrm{hr}} 200 \mathrm{mi} @ 50 \mathrm{mph}$ is $\frac{200 \mathrm{mi}}{4 \mathrm{hr}}$

$$
\frac{200 \mathrm{mi}}{5 \mathrm{hr}} \oplus \frac{200 \mathrm{mi}}{4 \mathrm{hr}}=\frac{400 \mathrm{mi}}{9 \mathrm{hr}}=44.4 \mathrm{mph}
$$

## A More Likely Scenario



## A More Likely Scenario

- Leave San Antonio.
- Stop in Schulenburg for pie and coffee.
- Drive to Beltway 8.
- Take Beltway 8 to I-45.
- Take I-45 to home.



## A More Likely Scenario

100 miles from San Antonio to Schulenburg. Average speed 60 mph .
Stop for pie and coffee. Average speed 0 mph .
70 miles to Beltway 8 (West).
Ayerage speed 70 mph .

200 miles

53.3 mph

20 miles from Beltway 8 to I- 45 .
Average speed 80 mph .
10 miles on I-45 to neighborhood.
Average speed 60 mph .
$\frac{100 \text { miles }}{100 \text { minutes }} \oplus \frac{0 \text { miles }}{40 \text { minutes }} \oplus \frac{70 \text { miles }}{60 \text { minutes }} \oplus \frac{20 \text { miles }}{15 \text { minutes }} \oplus \frac{10 \text { miles }}{10 \text { minutes }}$

## Puzzler:

A car is to travel two miles on a straight track.
It averages 30 mph for the first mile.
How fast will it need to go in the second mile to average 60 mph for the entire length?
$30 \mathrm{mph}=\frac{30 \mathrm{mi}}{60 \mathrm{~min}}=\frac{1 \mathrm{mi}}{2 \mathrm{~min}}$
$\frac{2 \mathrm{mi}}{2 \min }=\frac{1 \mathrm{mi}}{2 \min } \oplus \frac{1 \mathrm{mi}}{0 \mathrm{~min}}$


## Does Anybody Else Know This Stuff?

- Mathematicians
- Science Teachers

Accumulated Average
Average "builds" throughout term
Weights grades based on size

## Accumulated Average

Each assignment has its own total point value. Average $=\frac{\text { total points accumulated }}{\text { total points possible }}$
Rational Mean of the scores

$$
\frac{28}{30} \oplus \frac{46}{50} \oplus \frac{45}{45} \oplus \frac{56}{60} \oplus \frac{10}{15}=\frac{185}{200}=92.5 \%
$$

## The Answers:

## Question 3:





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## Answer:

## Question 3:



## What We've Learned So Far

- Fractions always represent parts of a whole.
- Two entirely different ways of combining them.
- One is the SUM.
- Other is the RATIONAL MEAN.
- Rational Means correctly calculate:
- Sports averages
- Average of two rates
- Combinations of sets or solutions
- Accumulated Grading
- When comparing fractions, we have to have CONTEXT.


## Why Weren't We Taught This?

- Curriculum largely fixed by $15^{\text {th }}$ Century.
- Rational Mean is recent.
- Pythagorean Means:
- Arithmetic Mean (Statistics) $\frac{a+b}{2}$
- Geometric Mean (Finance) $\sqrt{a b}$
- Harmonic Mean (Statistics, Eléctronics, Music) $\frac{2 d b}{(d+b b b}$
Rational Mean (Finance) $\frac{a+c}{b+d}$


## Are They Related?

## Rational us Harmonic <br> $$
\frac{2(40)(50)}{40+50}=\frac{4000}{90} \cdot 44 . \overline{4}
$$

Rational vs Arithmetic $\frac{50 \mathrm{mi}}{1 \mathrm{hr}} \oplus \frac{40 \mathrm{mi}}{1 \mathrm{hr}}=\frac{90 \mathrm{mi}}{2 \mathrm{hr}}$ When times are equal.
When denominators are equal. $\stackrel{B}{B} \frac{8}{\square}=\frac{B 1}{B}$

## The Big Picture

Combining fractions is more complicated than we imagined.

What - Now I have to teach TWO procedures for adding fractions?

NO!


## Procedures

- Rules that get the answer
- Memorized without understanding
- Don't know when to use them
- Misremember rule
- No idea of error
- Don't know what to do about it

The Psychology of Learning Mathematics

## Teaching for Understanding

- Two types of understanding
- Habit Learning (Procedural Learning)
"Rules without reason"
- Intelligent Learning (Conceptual Learning)
"Knowing what to do, and why"
- Most math instruction based on procedures


## Why?

-Often much easier to teach

- Immediate rewards
- Less knowledge involved, so answer comes more quickly
- Requires a new rule for each situation


## Conceptual Learning

- Takes longer to teach.
- Students sometimes resist it.
-Sómetimes involves more content.



## Example

$$
A=\frac{1}{2} b h
$$


$5 \times 56 \times 1=200$

## Extension



## Further Extension



## Conceptual Learning

- Takes longer to teach.
- Students sometimes resist it.
- Sómetimes involves more content.
- Easier to remember.
- More adaptable to new tasks.
- Is effective as a goal in itself.
- Leads to willingness to study further.


## Failure to teach Conceptually

- Use procedures inappropriately
- Can't transfer
- Baffled by word problems
- Math phobia
"I never could do math.".
"The negative attitude to mathematics, unhappily so common, even among otherwise highly-educated people, is surely the greatest measure for our failure and a real danger to our society."

Sir Herman Bondi,
Austian/British Mathematician and Cosmologist

A public unable to reason with figures is an electorate unable to discriminate between rational and reckless claims in public policy.

## Lynn Arthur Steen: Numeracy




## Teacher Comments

- The more they listened and spoke, the more they contributed, ... This was the best classroom atmosphere we have had so far this year.
- I've become much better at dealing with misconceptions. I now use them as great teaching opportunities.
- I am really pleased with how much more actively these pupils engage. They are much more prepared to volunteer for tasks, to develop their own methods, to listen and question.


## Pupil Comments

- I liked being able to argue and get my thoughts clear.
- I think I'm much better at sharing my ideas now. Everyone listens to each other's ideas, even the teacher!
- I liked being able to explain my ideas. I didn't expect everyone to want to know what I thought.


## What Should We Do?

- Start with real-world situations.
- Use manipulatives to teach concepts.
- Ask students to explain their thinking.
- Connect physical, pictorial and abstract.
- Different contexts, but the same concepts.


## Not a New Idea

## GENERAL VIEW OF THE PLAN.

Every combination commences with practical examples. Care has been taken to select such as will aptly illustrato the combination, and assist the imagination of the pupil in performing it. In most instances, immediately after the practical, abstract examples are placed, containing the same numbers and the same operations, that the pupil may the more easily observe the connexion. The instructer should be careful to make the pupil observe the connexion. After these are a few abstract examples, and then practical questions again.

## First Lessons in Arithmetic

Warren Colburn
A. 1. If you give $\frac{1}{2}$ of an orange to one boy, and $\frac{1}{4}$ to another, how much more do you give the first, than the second?
9. $\frac{1}{2}$ of an orange is how many $\frac{1}{4}$ of an orange?
3. If you give $\frac{1}{2}$ of an orange to one boy, and $\frac{1}{4}$ to another, how many $\frac{1}{4}$ would you give away? How many $\frac{1}{4}$ would you have left?
4. $\frac{1}{2}$ and $\frac{1}{4}$ are how many $\frac{1}{4}$ ?
5. A man gave to one labourer $\frac{1}{2}$ of a bushel of wheat, and $\frac{3}{4}$ to another; how many $\frac{1}{4}$ of a bushel did he give to both? How many bushels?
6. $\frac{1}{2}$ and $\frac{3}{4}$ are how many $\frac{1}{4}$ ? How many times 1 ?
7. A man gave $\frac{1}{2}$ of a barrel of flour to one man, and $\frac{3}{6}$ of a barrel to another; to which did he give the most?
8. $\frac{1}{2}$ is how many $\frac{1}{6}$ ?
9. A man bought $\frac{1}{3}$ of a bushel of wheat at one time, and $\frac{2}{6}$ of a bushel at another ; at which time did he buy the most?
10. $\frac{1}{3}$ is how many $\frac{1}{6}$ ?
11. A man bought $\frac{2}{3}$ of a yard of cloth at one time, and $\frac{4}{6}$ of a yard at another; at which time did he buy the most?
12. $\frac{2}{3}$ are how many $\frac{1}{6}$ ?
13. A man wished to give $\frac{1}{2}$ of a bushel of wheat


1a. $\frac{\mathbf{2}}{\mathbf{1 0}} \oplus \frac{\mathbf{4}}{\mathbf{1 1}}=$

2a. $\frac{\mathbf{2}}{9} \oplus \frac{\mathbf{1}}{11}=$

3a. $\frac{6}{12} \oplus \frac{8}{9}=$

4a. $\frac{4}{12} \oplus \frac{3}{12}=$

5a. $\frac{4}{7} \oplus \frac{3}{11}=$

6a. $\frac{4}{5} \oplus \frac{6}{7}=$
Hemarachcofethervopnolbanas, the fingn nombier slepresents the
Thaetforstofuanbac hetbretseintsitleed fuadHondxya Galgeosfencambleaniaris ischetlimecteonnof anmbetr its the frautidmoffarestherr bag of marbles Howisnmady inches of rain fell in Ifehtentomatysgogeeheoured into a bowl, what fraction of the marbles in the bowl will be red?

## Asking Questions

What "action" do you see in this problem?
Does it sound like something we' ve done before?
Can you use objects or draw me a picture?
Is this sometimes/always/never true?
How do you know that?
Do you see a pattern?
Can you explain that in your own words? What kind of answer should we expect?
Is this answer reasonable?

## Data Analysis

| Object | Circumference | Diameter | Ratio |
| :--- | :---: | :---: | :---: | :---: |
| Jar Lid | 32.6 | 10.2 | $\mathbf{3 . 2 0}$ |
| Bicycle Wheel | 210 | 66 | $\mathbf{3 . 1 8}$ |
| Coffee Lid | 68.6 | 20.2 | $\mathbf{3 . 4 0}$ |
| Mixing Bowl | 94.1 | 30 | $\mathbf{3 . 1 4}$ |
| Saucer | 57.5 | 17.9 | $\mathbf{3 . 2 1}$ |
| Dinner Plate | 82 | 26.5 | $\mathbf{3 . 0 9}$ |

What kind of "average" should we use here? Median? 3.19 Arithmetic Mean? 3.09
Qunacú © Geometic Mean? 3.20 Harmonic Mean? 3.20 Rational Mean? 3.20

## Fostering Analytic Thinking

Let students struggle with novel problems.

Sketch the graph of $f(x)=e^{\sin (0 x)}$ over a reasonable interval.


## Fostering Analytic Thinking

Sketch the graph of $f(x)=e^{\sin (0 x)}$ over a reasonable interval.

| $\boldsymbol{x}$ | $\sin (\mathbf{1 0 x})$ | $\boldsymbol{f ( x )}$ |
| :---: | :---: | :---: |
| 0 | 0 | 1 |
| 0.3142 | 0 | 1 |
| 0.6283 | 0 | 1 |
| 0.1571 | 1 | 2.7183 |
| 0.4712 | -1 | 0.3679 |
|  |  |  |
|  |  |  |
| $\frac{J \pi}{20}$ | -1 | $\frac{1}{e}$ |



## Tomorrow

9:45-11:15 am
Basic Student Understanding Through Effective Questioning
Karen Hardin
Right here in this room.

# NCTM Principles and Standards Reasonableness and Sense Making 

- Explain their thinking.
- Believe that mathematics makes sense.
- Make and evaluate mathematical conjectures and arguments.
- Appreciate the power of reasoning as a major part of mathematics.
- Construct proofs of mathematical assertions.


## Common Core State Standards

Clear understanding of what students are expected to learn.

Robust and relevant to the real world.

Fully prepare students for the future.

## CCSS Mathematical Tasks

Make sense of problems and persevere in solving them.
Reason abstractly and quantitatively.
Construct viable arguments and critique the reasoning of others.
Model with mathematics.
Use appropriate tools strategically.
Attend to precision.
Look for and make use of structure.
Look for and express regularity in repeated reasoning.

## Our Task

## To create a classroom where

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Give an education that will last a lifetime!


## Selected Resources

- Rice University School Mathematics Project
- rusmp.rice.edu
- Relational and Instrumental Understanding
- Prof. Richard R. Skemp
- http://www.blog.republicofmath.com/archives/654
- The Psychology of Learning Mathématics
- Prof. Richard R. Skemp, Amazon.com


## Selected Resources

- Knowing and Teaching Elementary Mathematics
- Liping Ma, Amazon.com
- First Lessons in Arithmetic
- Warren Colburn, Ámazon.com
- Why Johnny Can't Add: The Failure of the New Math
? $\rightarrow$ Morris Kline, Amazon.com


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SRedrce compround fraccians to sisig be oncs; mixed

## Thank you for your time.



$$
y \frac{4}{5}-\frac{3 q}{2} \quad 5 \quad \frac{3}{8}=5 \frac{5}{6} \operatorname{and} y=y
$$

$$
\begin{aligned}
& 39 \frac{15}{5} \text { and } 7 \\
& \begin{array}{l}
39 \times 56 \times 1=210 \\
10 \times 50 \\
7 \times 5 \times 56=1960 \\
5 \times 56 \times 1=200
\end{array}
\end{aligned}
$$



