Epistle #1
Encouraging Analytical and Predictive Thinking and Sense-Making in Mathematics Classrooms

Problem solving always seems to be one of the more difficult areas of school mathematics to teach, and for students to master. There are many reasons for this, of course; otherwise the task would be easily learned by almost every pupil under the guidance of a competent teacher.

One of the problems is comprehension. If students are poor readers or if the problem is stated in a language in which they are not proficient, the lack of comprehension of the problem situation is hindered before they ever get a chance to exercise their mathematical skills. (For such students, guessing is often the strategy they use, latching on to key words to tell them what to do.) Another problem may be a lack of familiarity with the context in which the problem is stated, or at the very least a complete lack of interest in the context by the student.

These obstacles can be overcome by a good teacher, especially with the assistance of learning specialists. But perhaps the hardest one to tackle is a student's difficulty with "if-then" thinking. This may come in the form of drawing a conclusion, getting the main idea, or predicting future events based on facts presented in the problem or story.

Student difficulty with inferential thinking is a recurring theme not only in the mathematics class, but in language arts, social studies, and science classes as well. Even the athletic program can suffer if student athletes cannot make a reasonable prediction of what their opponent is planning to do.

It is important, then, for all educators, and particularly mathematics teachers, to help their pupils learn the skills of thinking inferentially. A large part of this is accomplished if students are taught concepts rather than isolated rules. Doing so helps them gain a mindset that is always looking for patterns, itself a predictive action. But another component that is helpful is teaching students to analyze a problem before attempting to solve it. Pre-solving analysis allows students to gain "clues" about the eventual solution of the problem, as well as help them know that the answer they eventually obtain is reasonable for the problem presented. That is the subject of this epistle.

My middle daughter is a fifth-grade teacher in Humble ISD, and a technique her school uses (and I assume the other elementary schools in the district use as well) is called "Notice and Wonder." When faced with a problem situation, they often ask two questions: "What do you notice about this problem?" and "What do you wonder about it?" Each question is followed by a group-level brainstorming session (with no ideas rejected), allowing every member of the group to think about the problem, and listen to what others have to say about it. This technique works well in almost every subject they teach. (If you think about it, this type of thinking process is the foundation of the scientific method.) Guided by their teacher, students learn to pick out important information and use them to attack the problem at hand.

An interesting and novel context for this type of activity is the use of a poetic word problem. Poetic word problems sometimes appeared in the arithmetic texts in England and America, particularly in the eighteenth and early nineteenth centuries. They were often a type of puzzle, and usually appeared in what we would today call the miscellaneous section of the book (but which, at that time was known as the *promiscuous* set of problems – my how word meanings have changed in two hundred years)! This means that they were not tied to any particular section or topic in the book, and the learners had to draw on their knowledge to determine what skills the problem required. Because both the form and the language of the problem would be novel to the learners, this type of activity is recommended to be done as a whole-class problem-solving activity.

Here is such a problem, taken from an 1827 arithmetic text by Nathan Daboll and Samuel Greene.

If to my age there added be,
One-half, one-third, and three times three,
Six score and ten the sum will be;
What is my age, pray shew it me?

The first task to unpack is the language itself. Poetry allows, and often requires, a conciseness of wording and altering of word order to fit a given meter. Students can understand and overcome the latter, but might have difficulty comprehending the former. The teacher would probably need to explain the second line as meaning that half of the speaker's age, and a third of the speaker's age in years are added on to the original age, as well as an additional nine (three times three) years. The total is then claimed to be 130 years (six score and ten – the students are not likely to know the meaning of a score, unless they have read Lincoln's Gettysburg Address).

Once the "decoding" of the poetic form into prose is accomplished, the teacher can ask "If the extra nine years had not been added to the sum, what would the total have been? The students will arrive at the first step of the analysis of the problem when they say the total would be 121. "What do you notice about the number 121? Is there anything special about it?" The class will probably recognize it as 11×11. "What do you wonder about this total? Does this perhaps give us a clue about what the answer might be?" Students may or may not guess that the solution might involve a multiple of eleven, but they at least will notice that the sum is a whole number. The teacher might guide them toward it by noting that eleven is the only factor of this total besides 1, and the person is clearly not one year old. "Would we be surprised if the person's age was actually a multiple of eleven?" This is an important first step in making sense of the eventual solution.

"Now let's look again at the second line. Notice that, when half of the person's age is added to it, the answer is still a whole number. What kind of whole numbers have the property that half of them is also a whole number?" Of course, the students might point out that a third is also added, but the teacher might respond to this by saying something like “Let’s think about that observation for a minute. With your neighbor, try picking a few numbers – smaller ones, perhaps, to make the math easier – and see what happens if you try to add half of the number and a third of the number to it.” With a moment’s reflection, the students will realize that, for most of the numbers they choose, one (or both) of the actions result in a mixed-number answer, not a whole number one. But there are exceptions to this. “Did anyone find a number that stayed a whole number when you added half of it and a third of it to the number?” Students will realize that numbers like 6, 12, 18, and so on are the only ones that have this property. “Do you think this might be another clue to what the answer might be?” Students may say many things here, such as the answer must be an even number, or the answer must be a number three goes into.

At this point it would be a good time to recapitulate what the students have conjectured. “What clues do we now have about the answer?” The students should be able to point out that it besides being an even number, and a number that can be divided by three, it *might* also be a number that is divisible by eleven. With this information alone, students may use a guess-and-test strategy to determine that the smallest reasonable number that fits all the clues is 66.

Students who have been introduced to algebraic manipulations can be guided to write, and then solve, the linear equation $x + \frac{1}{2}x + \frac{1}{3}x + 9 = 130$. Regardless of the approach, they will have the ability to judge whether the solution they find makes sense.

If you would like another, more “modern” poetic age problem, consider this one I wrote in imitation of the problem just discussed. This one was fun to write, because it is a limerick, based on the name of a friend of mine.

The ages of Pam and her son
Are in ratio as 14 to 1.
But 11 years later,
She’ll be 3 times the greater.
Find their ages and this tale is done!

(The algebraic approach to this problem is, I believe, much harder than a guess-and-test approach, based on the “Notice and Wonder” technique discussed here, since it would involve setting up and solving a system of two equations in two unknowns.)

Note that the first clue toward finding a solution comes from the fact that Pam’s age is fourteen times as much as her son’s. This should lead the student to suppose that, whatever her age is, it must be a number that is divisible by both two and seven. The second comes from the fact that, after eleven years, her age is a multiple of three. Armed with this information, students again using guess-and-test as a strategy will quickly find the ages to be two and twenty-eight.

Keep in mind that the goal of these and similar activities is not the solving of trick or novel problems. The object is three-fold:

1. To teach students to look for clues for the solution *before* beginning to solve the problem;
2. To use the clues, they identify to make predictions about what the solution might look like; and
3. To be able to determine, based on their predictions, whether or not the solution they find is a reasonable one.

The acquisition and strengthening of these problem-solving skills will enhance the students' power and confidence as problem solvers. Students should therefore be given relatively frequent opportunities to exercise these skills throughout the school year. With practice, the teacher may find that the learners will become able to do more and more of the steps in this process on their own as their confidence increases.

One final note. Both students with algebraic experience and those who have not yet been exposed to them will benefit from participating in the foregoing type of activity. It will help both types of students to step back from the "what trick do I use here" mentality and direct them instead to develop the analytic and inferential skills that are so essential to every problem solver, and so often lacking in many of them.