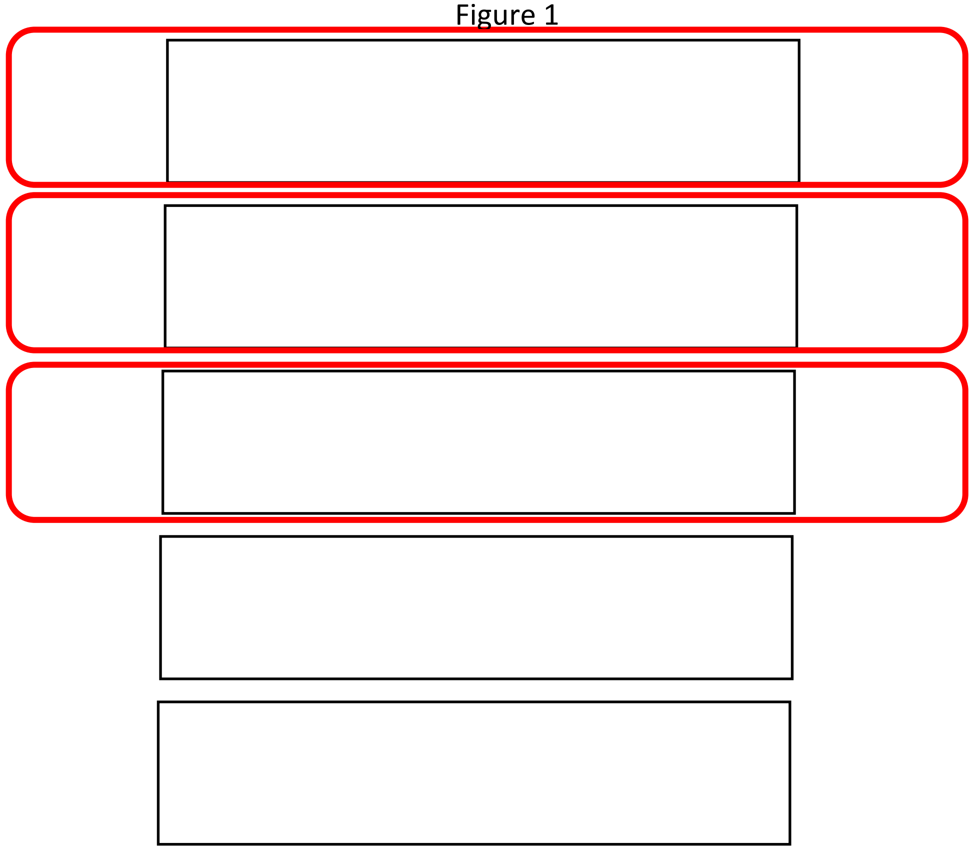
Epistle #2
Making Sense of Multiplying Fractions
Part One: Multiplying a Fraction by a Whole Number

A rule is a concise statement that tells how to find the answer to a problem. Most of the time, rules are the result of considerable experimentation by the people who devised them. The developers of a rule understand not only that it works, but why it works. But for someone who is simply given the rule to use, this is not the case. Students who are taught rules without this context and meaning know only that it works, but not why, or even what is going on in the process. Our brains are hard-wired to look for and devise rules and shortcuts, and when students are given the opportunity to do so, they will not only see *how* to use rules correctly, they will also understand why they work and what they are accomplishing. They will also be more confident that the answer they get makes sense.

To investigate multiplying fractions, it helps to begin with a problem that involves a whole number. Consider this problem: $\frac{1}{3} \times 5$. It is usually read as "one-third times five." It is important to emphasize here that it also can be read as "one-third of five." Connecting multiplying a number by a unit fraction to its associated division problem helps students see the interrelatedness of multiplication and division. Thinking of the problem through division, it can be represented in the following way. Since division can be seen as grouping or partitioning, an attempt to put five units equally into three groups would look something like what appears in Figure 1.

Figure 1



To place the remaining two units into the three groups, it is necessary to break up the two whole units. Ask students in pairs to discuss how this should be done. Various methods might be suggested, but the students will see that the simplest "breaking" would be into thirds, because this allows the "pieces" to be evenly distributed into three groups. (Incidentally, the word *broken* in this context is more appropriate than it might seem. The word "fraction" is derived from the Latin word *fractionem*, which means "broken." – think of a bone fracture – and early texts in English referred to fractions as "broken numbers.") See Figure 2.

Figure 2

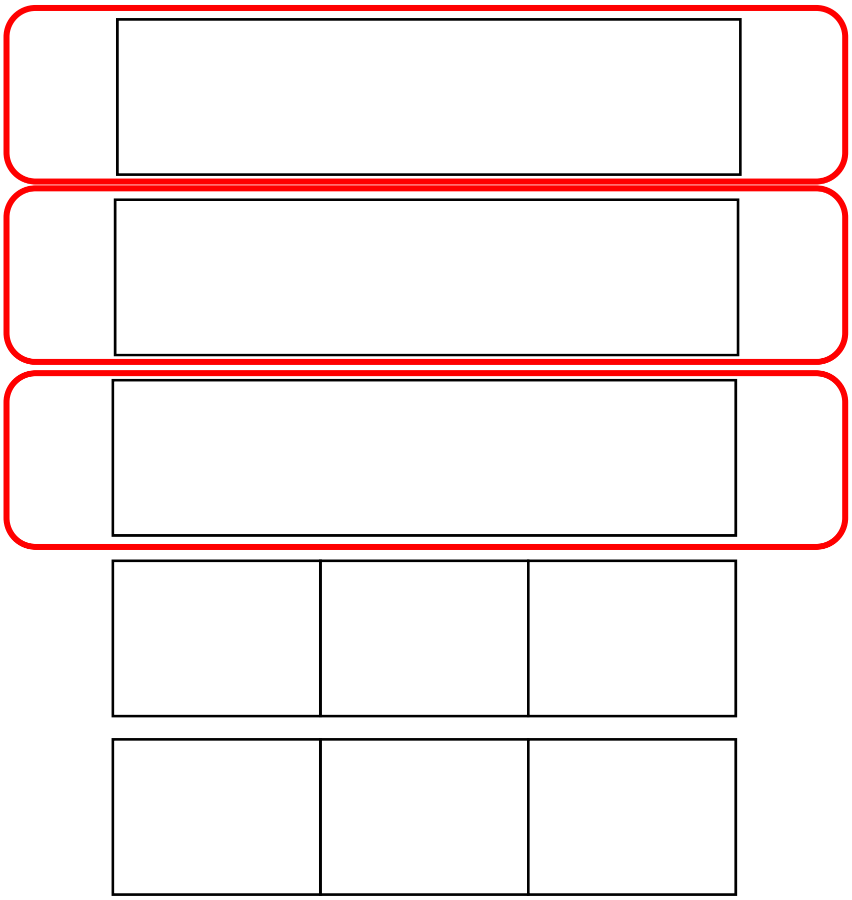
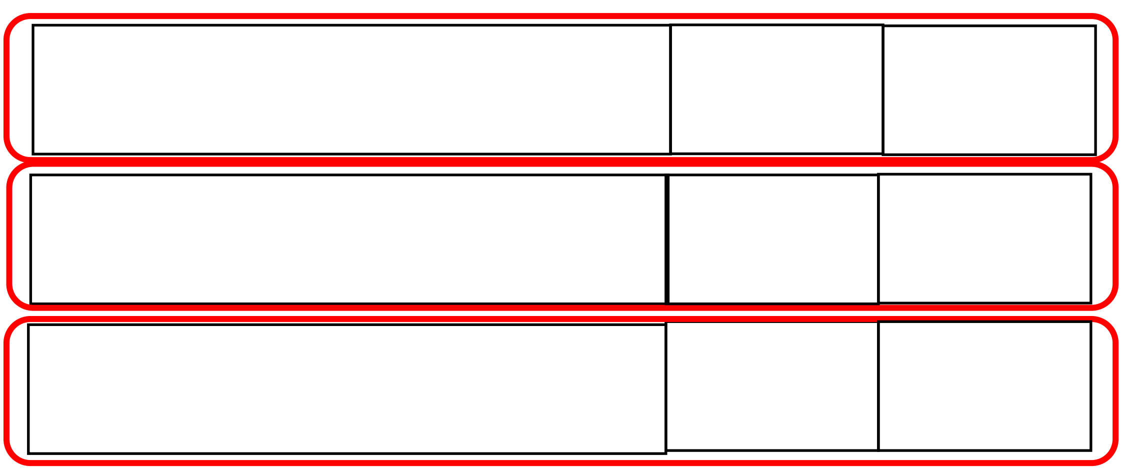


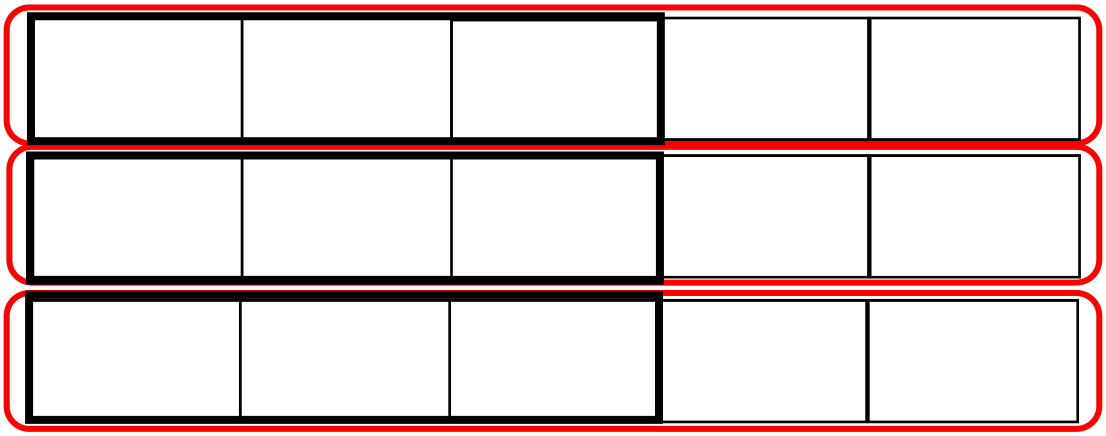
Figure 3 shows this once distributed.

Figure 3



How much is this? If the whole units are similarly partitioned, the students will see that each group contains $\frac{5}{3}$ parts. So $\frac{1}{3} \times 5 = \frac{5}{3}$. Because a total of $\frac{2}{3}$ was added to each group, the answer can also be stated as $1\frac{2}{3}$. See Figure 4.

Figure 4



At this point, it is important to reverse the factors and ask "What is $5 \times \frac{1}{3}$? Even though the students might have previously learned about the Commutative Property of Multiplication, it might not be at all obvious to them that it applies to fractional numbers as well. If a student offers "Won't that answer be the same?" Your response should be "Will it? Let's see." Figure 5 shows that the answer is indeed the same.

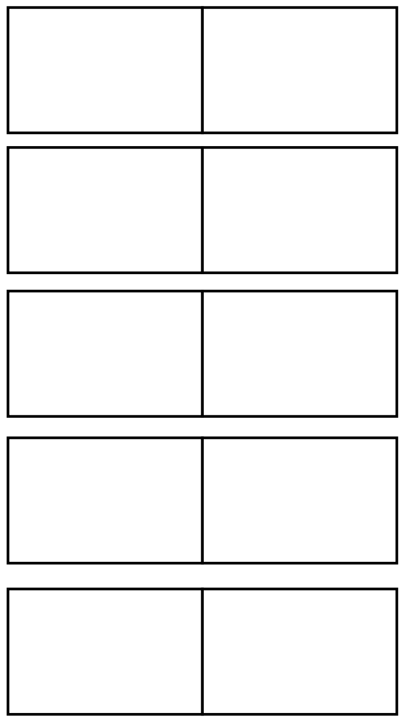
Figure 5

Work several more problems involving a unit fraction and a whole number, followed by its commutative version, until students are confident that, although the two versions ask something very different, the answers are the same. Point out that this gives them the choice to work the problem in the way that seems easiest to them, regardless of the way it was originally written. Be sure at this stage to include some problems like $\frac{1}{5} \times 5$, so that they see that the answer $\frac{5}{5} = 1$.

Now consider a problem like $\frac{2}{3} \times 5$, read "What is two-thirds of 5?" Ask students what is meant by the phrase "two-thirds." Sample answers might be "most of something" or almost all of something." In other words, we might guess that the answer is most of 5. Since this is not multiplication by a unit fraction, it does not appear to be one that can be directly related to whole-number division.

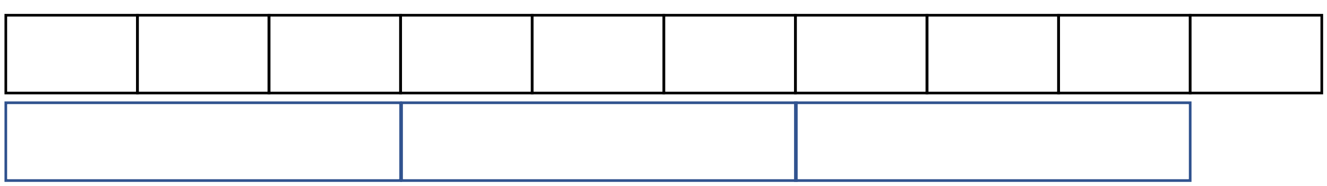
But we do know that the problem can be restated as $5 \times \frac{2}{3}$, because we have discovered that the Commutative Property of Multiplication applies to fractional numbers. This version is easy to see graphically. Using the same pieces as we did for the first problem, Figure 6 shows two-thirds, taken five times. This is $\frac{10}{3}$. So $5 \times \frac{2}{3} = \frac{10}{3}$.

Figure 6



If we place them, end-to-end, and compare them to some unit bars, we see that the total can also be written as $3\frac{1}{3}$, as shown in Figure 7.

Figure 7



But is this answer also two-thirds of 5? This step is a crucial one for the growth of students' analytical and sense-making skills. It helps them to see that the answer they get is sensible, and begins the development of their ability to reason through a problem without being wholly dependent upon a rote-learned algorithm.

Remind students that they have already seen that, for example, $5 \times \frac{1}{3} = \frac{5}{3}$, or $2 \times \frac{1}{7} = \frac{2}{7}$. "What, then, do we know about $2 \times \frac{1}{3}$?" Have students discuss this in pairs. Based on student sharing, the class should come to a consensus that $\frac{2}{3}$ is twice as much as $\frac{1}{3}$. "What can we conclude about $\frac{2}{3} \times 5$?" Again, small-group discussion would be helpful here, but it is important that they arrive at the conclusion that $\frac{2}{3} \times 5$ should be twice as much as $\frac{1}{3} \times 5$. "Since we already know that $\frac{1}{3} \times 5 = \frac{5}{3}$, what should be the answer to $\frac{2}{3} \times 5$?" The students should be able to agree that $\frac{10}{3}$ is a reasonable answer.

Ask several more questions of this type: "What is $\frac{1}{4} \times 7$?" "What is $\frac{3}{4} \times 7$?" until students are comfortable with transitioning from a unit fraction of a number and any fraction of the number. At this point, use examples that do not simplify. The key is to control all variables except the one on which you are working. Once students are able to solve these questions, including justifying them – with manipulatives whenever necessary – then it is a good time to work problems like "What is $\frac{1}{3} \times 6$?" and "What is $\frac{3}{4} \times 8$." When students answer "$\frac{6}{3}$," (and in the second case, "$\frac{24}{4}$," then ask "Do we have another name for this?" Use manipulatives, if necessary, to verify that $\frac{6}{3} = 2$ and $\frac{24}{4} = 6$."

The preceding activity shows students that concepts that they have learned are adaptable and open to expansion as new situations warrant. This flexibility is in contrast to rules, which are memorized without understanding, and are therefore inflexible and limited in their utility. Every new problem tends to require a new rule to fit it. In the next epistle, the concept of fraction multiplication will be expanded further to include that of a fraction by a fraction.