Epistle #3
Making Sense of Multiplying Fractions
Part Two: Multiplying a Fraction by a Fraction

In Epistle #2, two important concepts were developed:

1. The Commutative Property of Multiplication extends to the fractional numbers. This means that students are free to work whichever version of a problem that seems more sensible to them.
2. The relational nature of a unit fraction to its other proper fraction versions was explored. Once a solution for $\frac{1}{4}$ times a whole number is known, for example, the solutions to $\frac{n}{4}$ times that whole number can easily be found by multiplying the unit fraction solution by n.

In this epistle, the concept of multiplying any fraction by any other fraction will be developed.

Consider this product: $\frac{2}{3} \times \frac{1}{5} =?$ This product can be interpreted as the English sentence "What is two-thirds times one-fifth?" When the first factor in a multiplication problem is one or larger, the "×" sign is interpreted as "times" or "multiplied by." When the first factor is a fraction, it is usually interpreted as "of." So in this case, the product is read as "How much is two-thirds of one-fifth?" This leads to a simpler question "How much is one-third of one-fifth?" Students, based on the previous epistle, might already know the answer to this question. Not willing to depend on this, however, the following should be used to demonstrate the product. Figure 1 shows one whole unit divided into five parts, with one of them shaded to represent one-fifth.

Figure 1

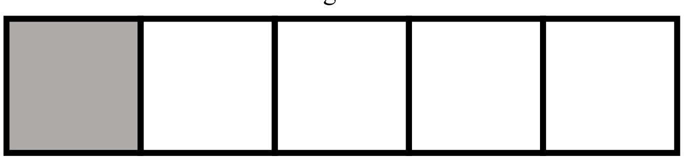
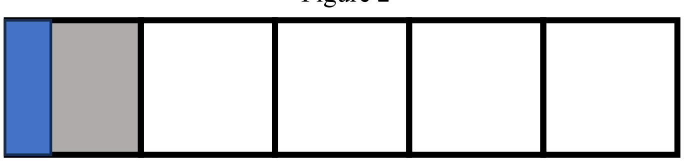


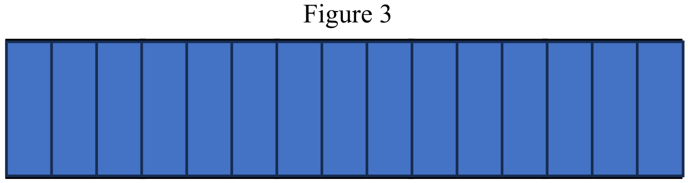
Figure 2 shows that shaded region, with one-third of it shaded differently.

Figure 2



How much of the whole unit does this smaller piece represent? If the remainder of the fraction bar is filled with duplicates of the smaller piece, as shown in Figure 3, we see that it will take fifteen of them to fill the whole. Therefore, each small piece is one-fifteenth of the whole.

Figure 3

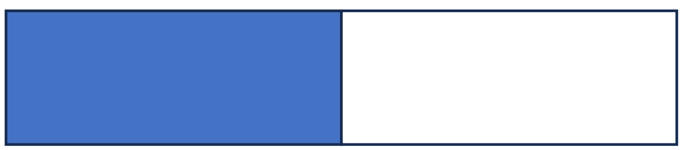


Hence, one-third of one-fifth is one-fifteenth, and two-thirds will be twice as much, or two fifteenths. Therefore, $\frac{2}{3} \times \frac{1}{5} = \frac{2}{15}$. Based on the second concept developed in Epistle #2, we also know that $\frac{1}{5} \times \frac{2}{3} = \frac{2}{15}$. Does this answer make sense? Ask students to discuss this in pairs. No matter which version of the problem we consider, we are asking to find a portion of a number that is already smaller than one. Therefore, the answer should be smaller than one as well. (In fact, the answer turns out to be *between* the two factors.)

This process should be repeated with carefully-chosen examples until students are confident of the result. Be sure that none of the examples would allow simplification of either the factors or the answer; all examples should involve numbers that are relatively prime. Once students are able to state the solution to a given problem, ask them to find the product with the factors reversed.

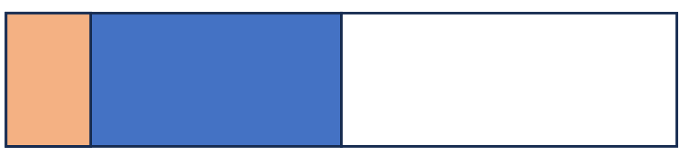
Let us look at one more example. Consider the statement $\frac{3}{4} \times \frac{1}{2} = ?$, which asks "How much is three-fourths of one-half?" Using the same process, Figure 4 shows a whole divided into two halves, with one of them shaded.

Figure 4



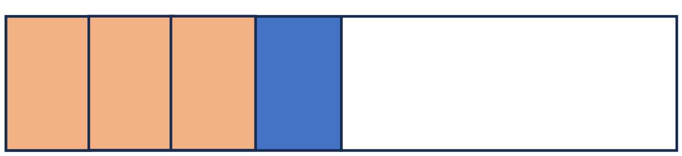
To answer the question, one must first ask "How much is one-fourth of one-half?" Figure 5 shows the one-half divided into fourths, with one of them shaded differently. It can be easily shown that this is one-eighth of the whole.

Figure 5



Three-fourths will then be three times as much, which is three-eighths, shown in Figure 6. We also know that one-half of three-fourths is also three-eighths.

Figure 6



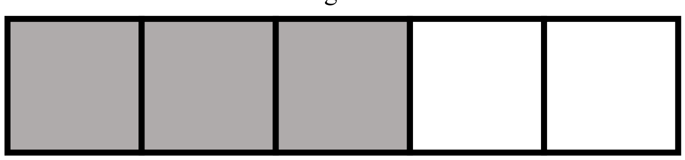
It is easy to see that the product, three-eighths, makes visual sense as well as conceptual sense. It is clearly three-fourths of the one-half, and it is also three-eighths of the whole, which students may verify for themselves.

Somewhere along the way, at least one student will see a pattern and ask about whether one can simply multiply the numerators together and the denominators together to get the same result. If this occurs, remember to respond with "Does it? Let's try another one, just to be sure." If no one makes this suggestion, the teacher can ask if anyone notices a pattern that might be a short-cut. In either case, the conjecture should be tested against more carefully-chosen problems, "just to be sure." If a student **does** see the pattern, name it after the student, such as "Sarah's Conjecture" or "Sarah's Rule." Explain that mathematics is full of rules that are named after the mathematician who found them. Thereafter, continue to say something like "Let's use Sarah's Rule to work this problem."

It is important that students see mathematics as the study of patterns. The teacher should say "Always expect to find a pattern. Be surprised if you don't." This encourages students to continually seek new patterns and look for ways to try to apply what they have learned to new situations.

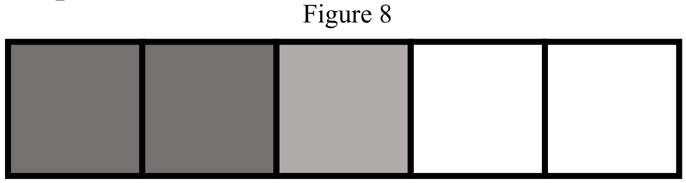
The next step is to introduce multiplication of fractions in which there are common factors in the numerators and denominators. The key once again is to use only problems in which a single number is available for simplification. More complicated situations may be used once students clearly understand how to simplify, and why. The first example might be $\frac{2}{3} \times \frac{3}{5}$. This asks for two-thirds of three-fifths. Three-fifths of a whole is shown as a shaded region in Figure 7.

Figure 7



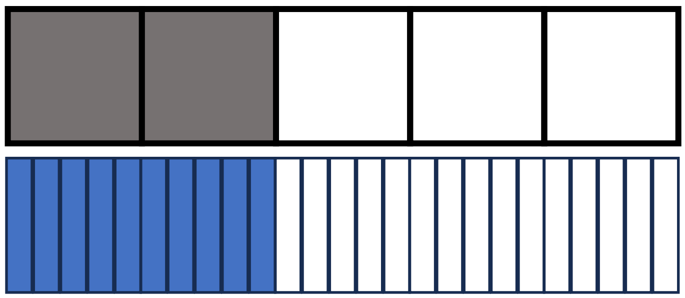
Ask students to discuss what portion of that shaded region would be $\frac{2}{3}$ of it. Two-thirds of that shaded region is shown in Figure 8 to be shaded darker. Since the region is already divided into three parts, they should be able to discern that two of those parts would represent two-thirds of the three-fifths.

Figure 8



Ask students to discuss what part of the whole this darker-shaded region represents. Since it is two parts of the five, the answer would be $\frac{2}{5}$. Therefore, $\frac{2}{3} \times \frac{3}{5} = \frac{2}{5}$. Ask students to solve the problem as in previous problems. They should get an answer of $\frac{6}{15}$. Figure 9 compares this with that shown in Figure 8.

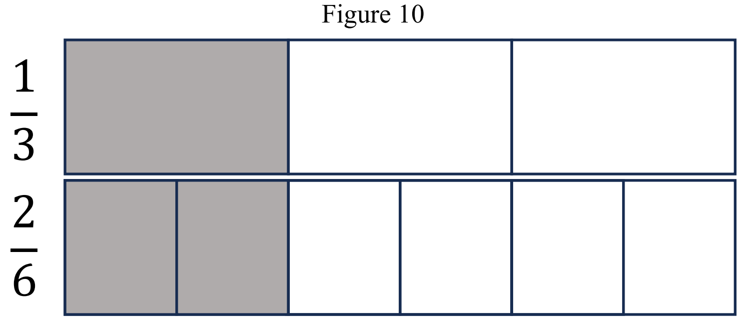
Figure 9



Ask students how the two darkly-shaded regions compare. How are they alike? How are they different? Sample responses might be that they are the same length, that the lower one is broken into more pieces, that one of them shows two-fifths, and the other six-fifteenths. Tell them that, when two fractions represent the same amount, we say that they are *equivalent* fractions. Explain that the word *equivalent* comes from Latin, which means "equal in value," because that is what they are. Although they look very different, they both show the same fraction of the whole, so $\frac{2}{5}$ is equivalent to $\frac{6}{15}$. Point out that people often prefer $\frac{2}{5}$ as the answer, because its numerator and denominator are smaller numbers than 6 and 15. In fact, no other fraction can use smaller whole numbers to represent this amount, so people say that $\frac{2}{5}$ is in *simplest form*. (Teachers used to say that the fraction was "reduced" or "reduced to lowest terms." This is misleading. Although the *numbers* in the numerator and denominator are smaller, the values of the fractions are the same.)

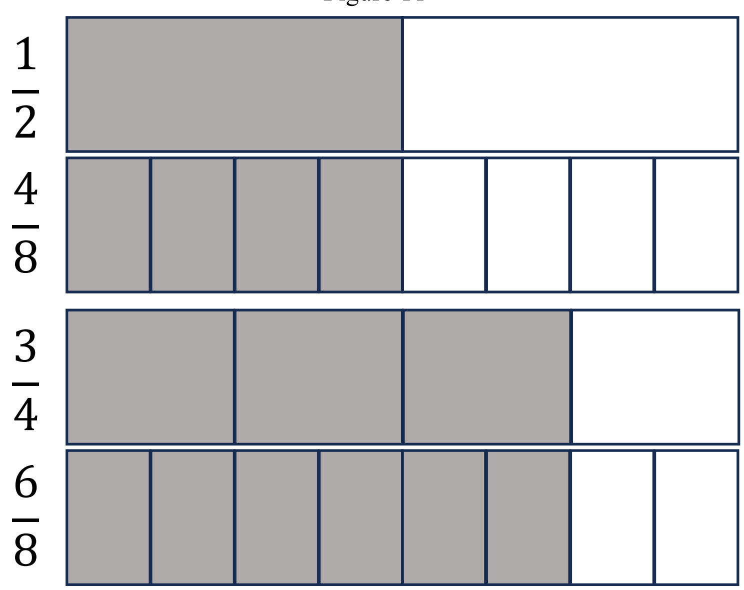
Now show students the fractions in Figure 10.

Figure 10



What do you notice about these two fractions? Let students discuss this, and then report their findings. Responses might be that they both have the same amount shaded, although the shaded regions are broken up differently. Someone might even say that both fractions are equivalent. Ask them to look again at the fraction bars in Figure 9 as well as the ones in Figure 10. “Do you notice any relationship between the fraction bars and the names of the fractions? Do you see any patterns?” Remind students that the numerator tells how many parts are in the fraction, and the denominator tells how many parts are in the whole. The relationship they need to discern is that the shaded region is the value of the numerator and the total number of parts in the whole is the value of the denominator. In Figure 9, there are three times as many parts in the darkly-shaded region of $\frac{6}{15}$ as there are in $\frac{2}{5}$. In Figure 10, there are twice as many shaded parts in $\frac{2}{6}$ as there are in $\frac{1}{3}$. In Figure 9, if the number of parts in the shaded region, as well as the total number of parts in the whole are divided by three, the resulting fraction will be identical to two-fifths. Similarly, in Figure 10, if the number of parts in the fractional part as well as the whole are divided by two, the result will be identical to one-third. This is what is meant be simplifying a fraction – eliminating a factor that is common to both the numerator and the denominator. Give students additional examples to consider and ask them to determine what would need to be divided out in each case to make the two fractions identical. Some additional examples are shown in Figure 11, but the teacher should feel free to choose any examples that have a single common factor in them.

Figure 11



Students should see that if the lower fraction of each pair of fraction bars has a common number divided out (4 in the first pair, and 2 in the second), the result will be identical to the upper fraction of the pair.

Tell the students "When dividing out a common factor to simplify a fraction, it is important to show to others that this has been done, so that those who are reading your work can see what you have done. The way this is done is by showing the result of the division on the fraction itself in this way. For example, here is how to show that four-eighths has been simplified to one-half." Show them Figure 12 and explain how it demonstrates the simplification process, saying "4 divided by 4 is 1, and 8 divided by 4 is 2."

Figure 12

$$\frac{\cancel{4}^{1}}{{}_{2}\cancel{8}} = \frac{1}{2}$$

Practice this with as many examples as needed to ensure that students can:

1. State the common factor that needs to be divided out of the numerator and denominator,
2. Explain the process through the division, and
3. Write the simplified result.

At this point say "Let's look again at $\frac{2}{3} \times \frac{3}{5}$. "What would this problem look like if we divided out the common factor?" Students might ask if it is OK to divide out common factors from two different fractions. Say "If we did, what would it look like?" Students should be able to show work like that shown in Figure 13.

Figure 13

$$\frac{2}{{}_{1}\cancel{3}} \times \frac{\cancel{3}^{1}}{5} = \frac{2}{5}$$

Say "Is this the result we got earlier?" The students will agree. (Perhaps another student might get the naming honor this time.) Practice a few more problems like this before continuing.

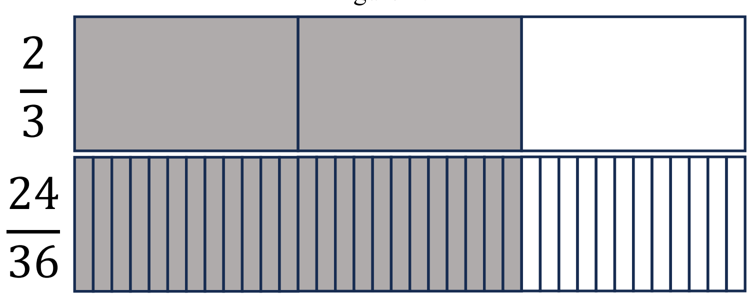
Now that students understand the process of simplifying out a single common factor, it is time to progress to problems that contain more than one common factor, such as $\frac{3}{4} \times \frac{8}{9}$. "What do you notice about this problem?" "What do you wonder about it?" Let them discuss it together. They will probably notice that there are two pairs of numbers that could be simplified, so they might ask if it is OK to divide out more than one set. Say "Let's see." "Based on what we already know, what would the product be?" They should be able to determine that the product would be $\frac{24}{36}$. "Do the numerator and denominator of this fraction have any common factors?" Students will find four answers to this: 2, 4, 6, and 12. "Which of these would be the best one to use?" They should be able to agree that 12 is the best answer. "Now simplify $\frac{24}{36}$ by dividing the numerator and denominator by 12." They should get $\frac{2}{3}$. "Now suppose we try dividing out any common factors in the original problem and see what we get." Figure 14 shows this.

Figure 14

$$\frac{{}^{1}\cancel{3}}{{}_{1}\cancel{4}} \times \frac{\cancel{8}^{2}}{\cancel{9}_{3}} = \frac{2}{3}$$

"Is the result the same?" They should agree that this is the case. Figure 15 shows the comparison of $\frac{24}{36}$ and $\frac{2}{3}$. This should verify the correctness of their thinking.

Figure 15



The conceptual framework that was built in this epistle allows students to connect the procedural process of multiplying a fraction by a fraction, both with and without common factors, in a way that students know how to use the standard algorithm, are able to see the physical representation of each problem when needed, and can verify that the result they get makes sense. This concept is now available to students when they encounter multiplication of mixed numbers with fractions, not as a brand-new rule to memorize, but as an extension of what they already understand about how to multiply fractions. This skill will be extended even further when they are shown the multiplication of decimals – which are, in fact, decimal *fractions*. Their understanding of decimal multiplication will be seen as just another notational method of multiplying fractions with certain special characteristics, and not totally different numbers that have no relation to what they already understand.