

# Geometry Module 



Funding for the Geometry Module was provided by the Texas Education Agency and the Texas Higher Education Coordinating Board.

# The Texas Education Agency and the Texas Higher Education Coordinating Board Geometry Module 

## Introduction

The Rice University School Mathematics Project (RUSMP) developed the Geometry Module as a comprehensive teacher training module with funding from the Texas Education Agency and the Texas Higher Education Coordinating Board. The Geometry Module effectively assists teachers in developing a deeper understanding of the underlying concepts that support the Texas Essential Knowledge and Skills (TEKS) in Geometry and helps teachers develop the pedagogical tools necessary to provide their students the opportunity to meet Texas' challenging state content and student performance standards. The Geometry Module also supports related TExES Mathematics Competencies. The rigor of the Geometry Module is of sufficient nature as to allow participating teachers who have not yet met the requirements of a "highly qualified" teacher, as defined by the United States NO CHILD LEFT BEHIND ACT of 2001 (NCLB), to progress towards this goal.

## Theoretical Framework for the Geometry Module

The National Council of Teachers of Mathematics (NCTM) proposed major changes in pre-college mathematics curriculum in its Standards (1989, 1991, 1995, 2000). The National Research Council in Adding It Up: Helping Children Learn Mathematics (2001) and Educating Teachers of Science, Mathematics, and Technology: New Practices for the New Millennium (2001) provides researchbased recommendations for teaching and learning that support effective mathematics education. This research indicates that active, student-centered mathematical investigations, group cooperation, and alternative assessments are more effective in reaching diverse student populations than the passive, teacher-centered learning methods which have dominated mathematics instruction in the past. The Geometry Module materials are consistent with these recommendations.

The Geometry Module is based on the van Hiele model of geometric thought. NCTM in its Standards (1989), acknowledged the importance of the van Hieles' research.

Development of geometric ideas progresses through a hierarchy of levels. Students first learn to recognize whole shapes and then to analyze the relevant properties of a shape. Later they can see relationships between shapes and make simple deductions (p. 48).

Traditional geometry curriculum often fails, because there is a mismatch between geometry instruction and a student's van Hiele level. The hierarchy of levels in the van Hiele model consists
of (1) the Visual Level, (2) the Descriptive Level, (3) the Relational Level, (4) the Deductive Level, and (5) Rigor. The Geometry Module provides van Hiele-based experiences (Crowley, 1987) to move participants through the hierarchy from the Visual Level to Rigor. The Geometry Module provides descriptive behavior criteria which identify the different van Hiele levels of student performance, so that participants may identify and select corresponding activities to ensure success for all. Throughout the Geometry Module, participants will identify the van Hiele levels within the activities.

## Tools for Learning Geometry

The Geometry Module utilizes construction tools, manipulatives, and technology: (1) to address various learning styles, (2) to model or represent mathematical concepts, (3) to abstract from the manipulative representations, (4) to construct and explore mathematical properties of geometric objects, (5) to generate authentic data, and most importantly (6) to progress participants through the van Hiele levels. The appropriate use of construction tools, manipulatives, the graphing calculator, The Geometer's Sketchpad, and NonEuclid is incorporated into module materials.

## RUSMP's Unique Qualifications to Write the Geometry Module

RUSMP was established in 1987, with a grant from the National Science Foundation (NSF), in order to provide a bridge between the Rice University mathematics research community and Houston-area mathematics teachers. In addition to the original grant, RUSMP has received funding from a second NSF grant, the United States Department of Education Eisenhower and Teacher Quality Programs, and from corporations, foundations and school districts. The mission of RUSMP is to help teachers and administrators better understand the nature of mathematics, the effective teaching and assessing of mathematics, and the importance of mathematics in today's society. RUSMP's major goal is to enhance the mathematical and pedagogical knowledge of Houston PreK12 teachers and support them in implementing more effective mathematics programs.

The RUSMP approach is founded on the belief that sustained instructional changes can best be supported through the development of professionalism among teachers and the creation of a network of teachers who have extensive knowledge of both mathematical content and pedagogy. All RUSMP activities are designed to support the development of teachers' professionalism. RUSMP has developed an extensive array of programs and courses available to teachers and administrators. These include long-term, intensive professional development for teachers, day-long workshops, and opportunities for networking across schools and districts. In addition, RUSMP has
undertaken several collaborative projects with districts, schools, and other community members in the Houston area. While there is great diversity among the programs and activities offered by RUSMP, they are all anchored by a common curriculum and approach to instruction. The Geometry Module is the latest of RUSMP's efforts to improve the teaching of pre-college mathematics.

As a result of RUSMP's eighteen-year partnership with Houston-area school districts to improve mathematics instruction, RUSMP has the knowledge and experience necessary to develop an effective Geometry Module that meets the needs of current and future teachers. The Geometry Module builds upon the strengths and recommendations of prior curricula that RUSMP has designed and implemented for Houston-area PreK-12 teachers.

## Acknowledgements

Funding for the Geometry Module was provided by the Texas Education Agency and the Texas Higher Education Coordinating Board. The Geometry Module was developed under the direction and with the assistance of:

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January 28, 2004
Anne Papakonstantinou
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Dear Anne,

It was a pleasure seeing you last week. As we discussed, Key Curriculum Press is pleased to support you in your and your colleagues' efforts in producing geometry curriculum and materials in support of the Department of Higher Education, Participation \& Success -- Institution \& Educator Initiatives. Accordingly, we extend permission to you to reproduce, in print or electronically, portions of text and diagrams from the following Key Curriculum Press Publications for inclusion in Initiative materials:

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Please don't hesitate to contact me directly at (800) 338-7638, extension 124 or via email at ktaylor@keypress.com if there are questions or if I can be of assistance.

Sincerely,
Kelvin Taylor

April 27, 2004

Dear Anne:

You are permitted to load The Geometer's Sketchpad® software program onto the computer labs in Fort Worth and Houston for the duration of the Geometry Module Workshops. After the completion of the workshops, please unload the programs from all the computers.

Feel free to contact me if you have any questions or concerns.
Thank You,

Lesa Zimmerman
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# The Texas Education Agency and The Texas Higher Education Coordinating Board Geometry Module 

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## Comprehensive materials list

## Consumables

easel paper (several sheets per group of 4)
colored markers
patty paper (several pieces per participant)
graph paper
colored pencils (1 package of assorted colors per group of 4)
centimeter grid paper (several sheets per participant)
small colored dot (1 for demonstration)
transparency sheet (several sheets per group of 4 and 1 for demonstration)
overhead projector pens ( 1 package of 4 colors per group of 4 )
unlined 8.5 in . by 11 in . paper (several sheets per person)
3 in. by 5 in. index cards ( $11 / 2$ card per participant)
11 in . by 17 in . paper ( 1 per participant)
masking tape ( 1 roll per group of 4 )
cardstock
floral wire (several pieces per participant)
modeling clay
one-inch easel grid paper ( 1 sheet per group of 4)
spaghetti
clear tape (1 roll per group of 4)
glue (1 bottle or stick per table)
cups (preferably large plastic cups)
geoboard dot paper (several sheets per participant-provided in the Appendix)
3 in. square adhesive notes ( 2 of different colors for each participant)
paper cone shaped drinking cups
plastic rice
string ( 1 spool per group of 4)
equilateral triangle paper with side length at least one inch (several sheets per participant provided in the Appendix)

## Non-consumables

selection of geometry reference books or textbooks
centimeter ruler (1 per participant)
protractor (1 per participant)
linking cubes (several per participant)
small object such as a color tile
plastic mirror (1 per participant)
compass (1 per participant)
graphing calculator (1 per participant)
scissors (1 pair per participant)
straightedge (1 per participant)
centimeter cubes
geoboard (1 per participant)
centimeter grid transparency ( 1 for demonstration)
flexible protractor (1 per participant)
globe, beach ball, or Lénárt sphere (1 per group of 4)
transparencies "Constructing a Polygon's Exterior Angles" and "Determining the Sum of a Polygon's Exterior Angles"
wire-frame constructions from Unit 2: Exploring Prisms

## Technology

PowerPoint presentation: The van Hiele Model of Geometric Thought (or transparencies of Power Point slides)
Flash animation video 3-D.html
The Geometer's Sketchpad with sketches: Dilation Investigation, Mona Lisa, Golden Construction, Spiral, Trigonometry Ratios, Trigonometry Tracers
Computer lab and/or computer with projector
NonEuclid (available at http://cs.unm.edu/~joel/NonEuclid/NonEuclid.html)

## Required Materials by Activity

| Unit 1 - Transformations |  |
| :--- | :--- |
| Activity Name | Materials |
| Terms and Definitions | a selection of geometry reference books or <br> textbooks, easel paper, colored markers |
| What is a Translation? | easel paper, centimeter ruler, colored markers |
| Reflections | easel paper, centimeter ruler, colored markers, <br> patty paper, graph paper |
| Theoretical Framework: The van <br> Hiele Model of Geometric <br> Thought | PowerPoint presentation: The van Hiele Model of <br> Geometric Thought or transparencies of <br> PowerPoint slides |
| Rotations | centimeter ruler, patty paper, protractor, colored <br> pencils, centimeter grid paper, small colored dot, <br> transparency sheets (1 per group of 4), two <br> overhead pens of different colors (for each <br> group) |
| Composite Transformations | protractor, centimeter ruler, transparency sheets <br> (1 per group of 4), overhead projector pens in at <br> least two different colors |
| Tessellations | centimeter ruler, patty paper, protractor, colored <br> pencils, unlined 8.5 in. by 11 in. paper, 3 in. by 5 <br> in. index card cut in half, 11 in. by 17 in. sheet of <br> paper, colored markers, masking tape, easel <br> paper |
| Do You See What I See? | linking cubes, plastic mirrors, a small object such <br> as a color tile for each participant |


| Unit 2 - Triangles |  |
| :--- | :--- |
| Activity Name | Materials |
| Equilateral Triangles | patty paper, straightedge, compass, easel paper, <br> colored markers |
| Two Congruent Angles | patty paper, straightedge, protractor, compass, <br> easel paper, colored markers |
| Scalene Triangles | patty paper, centimeter ruler, compass, protractor |
| The Meeting Place | patty paper, centimeter ruler, compass, calculator |


| Unit 3- Quadrilaterals |  |  |
| :---: | :---: | :---: |
| Activity Name Materials |  |  |


| Isosceles Right Triangle <br> Reflections | colored pencils, easel paper, colored markers, <br> centimeter ruler, transparency |
| :--- | :--- |
| Scalene Right Triangle Reflections | colored pencils, easel paper, graph paper, colored <br> markers, centimeter ruler |


| Scalene Acute/Obtuse Triangle <br> Reflections | colored pencils, easel paper, colored markers, <br> centimeter ruler |
| :--- | :--- |
| Rotate a Triangle | easel paper, graph paper, colored markers, patty <br> paper, centimeter ruler |
| Truncate a Triangle's Vertex | easel paper, graph paper, colored markers, <br> centimeter ruler |
| Vesica Pisces | compass, easel paper, colored markers, <br> centimeter ruler |
| Exploring Prisms | cardstock, scissors, floral wire, modeling clay, <br> one-inch grid easel paper, Flash animation video <br> 3-D.html, computer lab and/or computer with <br> projector |


| Unit 4 - Informal Logic/Deductive Reasoning |  |
| :--- | :--- |
| Activity Name | Materials |
| Informal Logic | easel paper, colored markers |
| Inductive Triangle Congruence | unlined 8.5 in. by 11 in. paper, compass, <br> centimeter ruler, protractor, spaghetti, scissors |
| Deductive Triangle Congruence |  |
| Quadrilateral Proofs | easel paper, colored markers |
| Alternate Definitions | easel paper, colored markers |
| Circle Proofs |  |


| Unit 5 - Area |  |
| :--- | :--- |
| Activity Name | Materials |
| What Is Area? | 3in. by 5 in. index cards, patty paper, <br> straightedge |
| Investigating Area Formulas | transparency sheets, colored pencils, glue or clear <br> tape, patty paper, scissors |
| Area of Trapezoids | patty paper, scissors |
| Area of Circles | cups (preferably large plastic cups), glue or clear <br> tape, graphing calculator, colored markers, patty <br> paper, scissors |
| Applying Area Formulas | graphing calculator |
| What Is Surface Area? | centimeter grid paper, linking cubes, scissors, <br> straightedge, tape |


| What Is Volume? | centimeter cubes, straightedge, centimeter grid <br> paper, scissors, tape |
| :--- | :--- |
| Net Perspective | paper, scissors, tape, rulers, centimeter grid paper <br> (optional), centimeter cubes |
| Area Proofs | colored markers, easel paper |


| Unit 6 - Pythagoras |  |
| :--- | :--- |
| Activity Name | Materials |
| Sides of Squares | centimeter grid paper |
| Squares on the Sides of Acute or <br> Obtuse Triangles | centimeter grid paper, centimeter ruler |
| Applying Pythagoras, Part I | graphing calculator |
| Pythagorean Triples | transparencies of the tables for the activity, <br> calculator |
| Special Right Triangles | geoboard or geoboard dot paper (provided in the <br> appendix), unlined 8.5 in. x 11 in. paper |
| Distance Formula | centimeter grid paper, centimeter grid <br> transparency, 3 in. square adhesive notes in two <br> colors (one of each color per participant) |
| Applying Pythagoras, Part II | graphing calculator |


| Unit 7 $\mathbf{-}$ Polygons and Circles |  |
| :--- | :--- |
| Activity Name | Materials |
| Diagonals of a Polygon | straightedge, graphing calculator |
| Interior and Exterior Angles of a <br> Polygon | graphing calculator, straightedge, unlined 8.5 in. <br> by 11 in. paper, scissors, tape, transparencies <br> "Constructing a Polygon's Exterior Angles" and <br> "Determining the Sum of a Polygon's Exterior <br> Angles" |
| Polygons in Circles | graphing calculator, centimeter ruler |
| Angles Associated with a Circle | protractor, centimeter ruler |
| Parts of a Circle | compass, centimeter ruler, graphing calculator, <br> easel paper, colored markers |


| Unit 8 - Similarity |  |
| :--- | :--- |
| Activity Name | Materials |
| Magnification Ratio | graphing calculator, compass, centimeter grid <br> paper, protractor or patty paper, straightedge |
| What Do You Mean? | compass, centimeter grid paper, patty paper, <br> centimeter ruler |

Dilations | lompass, The Geometer's Sketchpad, The |
| :--- |
| Geometer's Sketchpad Sketch: Dilation |
| Investigation, centimeter grid paper, straightedge |

| Similarity and the Golden Ratio | compass, The Geometer's Sketchpad, The <br> Geometer 's Sketchpad Sketches: Mona Lisa, <br> Golden Construction, Spiral, graphing calculator, <br> centimeter grid paper, patty paper, straightedge |
| :--- | :--- |
| Trigonometry | cardstock, compass, The Geometer's Sketchpad, <br> The Geometer's Sketchpad Sketches: <br> Trigonometry Ratios, Trigonometry Tracers, <br> graphing calculator, centimeter grid paper, patty <br> paper, protractor, scissors, straightedge |
| Exploring Pyramids and Cones | wire-frame constructions from Unit 2: Exploring <br> Prisms, centimeter ruler, scissors, protractor, <br> patty paper (optional), compass (optional), paper <br> cone-shaped drinking cups, plastic rice, cardstock |


| Unit 9 - Non-Euclidean Geometries |  |
| :--- | :--- |
| Activity Name | Materials |
| $\begin{array}{l}\text { When is the Sum of the } \\ \text { Measures of the Angles of a } \\ \text { Triangle Equal to } 180^{\circ} \text { ? }\end{array}$ | $\begin{array}{l}\text { straightedge, compass, patty paper, colored pencils, } \\ \text { transparency sheet, scissors, overhead projector } \\ \text { pens }\end{array}$ |
| $\begin{array}{l}\text { Euclid's First Five Postulates in } \\ \text { Euclidean Space }\end{array}$ | straightedge, protractor |
| $\begin{array}{l}\text { Curvature in Different } \\ \text { Geometries }\end{array}$ | $\begin{array}{l}\text { Euclid's First Five Postulates in } \\ \text { Elliptic Space }\end{array}$ |
| $\begin{array}{l}\text { Euclid's First Five Postulates in } \\ \text { Hyperbolic Space }\end{array}$ | $\begin{array}{l}\text { flexible protractor, string, overhead projector pens, } \\ \text { globe, beach ball, or Lénárt sphere }\end{array}$ |
| lab and/or computer with projector, NonEuclid |  |
| (available at |  |
| http://cs.unm.edu/~joel/NonEuclid/NonEuclid.html) |  |$]$

Suggested Timeline

## The Texas Education Agency and the Texas Higher Education Coordinating Board Geometry Module Suggested Institute Timeline

This suggested timeline assumes 10 days of instruction with $\mathbf{6}$ hours of instruction per day.

Day 1
Hour 1 Welcome and Pre-Test
Hours 2-5 Unit 1: Introduction and Transformations
Hour 6 The Geometer's Sketchpad Unit 1: Introduction to the Program
Day 2
Hours 1-2 Unit 1: Introduction and Transformations (cont.)
Hours 3-5 Unit 2: Triangles
Hour 6 The Geometer's Sketchpad Unit 2: Transformations
Day 3
Hours 1-2 Unit 2: Triangles (cont.)
Hours 3-6 Unit 3: Quadrilaterals
Day 4
Hours 1-4 Unit 3: Quadrilaterals (cont.)
Hours 5-6 The Geometer's Sketchpad Unit 3: Triangles and Quadrilaterals
Day 5
Hours 1-5 Unit 4: Reasoning
Hour 6 Unit 5: Area
Day 6
Hours 1-5 Unit 5: Area (cont.)
Hour 6 The Geometer's Sketchpad Unit 4: Perimeter and Area
Day 7
Hours 1-5 Unit 6: Pythagoras
Hour 6 The Geometer's Sketchpad Unit 5: Pythagoras
Day 8
Hours 1-5 Unit 7: Polygons and Circles
Hour 6 The Geometer's Sketchpad Unit 6: Polygons and Circles
Day 9
Hours 1-6 Unit 8: Similar Figures and Trigonometry (The Geometer's Sketchpad embedded)

Day 10
Hours 1-5 Unit 9: Non-Euclidean Geometries (NonEuclid embedded)
Hour 6 Post-Test and Closing

## Unit 1 - Transformations

## Terms and Definitions

Overview: This activity establishes a common language of terms and definitions which will be used throughout the module.

## Objective: TExES Mathematics Competencies

III.012.A. The beginning teacher understands axiomatic systems and their components (e.g., undefined terms, defined terms, theorems, examples, counterexamples).
V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

## Geometry TEKS

b.1.A. The student develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems.

Background: No specific geometry knowledge is needed.
Materials: a selection of geometry reference books or textbooks, easel paper, colored markers

New Terms: This activity creates the glossary of basic terms which will be used throughout the module.

## Procedures:

Distribute the activity sheet. Participants use available reference materials to establish the common terminology and symbol usage for the terms listed. This is the beginning of a glossary to be developed during the module. Terms, definitions, and conjectures which arise during the module can be added to the glossary.

The activity, which should take no longer than 30 minutes, can be divided up so that each group defines a different set of terms. The terms have been grouped for this purpose. Provide easel paper and markers if you wish to display the terms for whole class reference and correction if needed.

The following four terms are undefined in the Euclidean axiomatic system. Describe each term in your own words.

A point has no size. It has only location in space. It is represented with a dot and named with a capital letter.

A line is a straight, continuous arrangement of infinitely many points. It is infinitely long and extends in two directions but has no width or thickness. It is represented and named by any two distinct points that lie on it.
A plane has length and width but no thickness. It is a flat surface that extends infinitely along its length and width. It can be named with a script capital letter, such as $\mathbb{P}$.
Space has length, width and thickness, extending infinitely along all three dimensions.
Define and/or draw representations of the following terms using conventional symbols. Possible definitions are provided below. Participants should include diagrams showing conventional symbols, such as matching tick marks for congruent figures, matching arrows for parallel lines and right angle and perpendicular signs.
I.

Collinear points lie on the same line.
Coplanar points or lines lie on the same plane.
A line segment is that part of a line that consists of two points, called endpoints, and all the points between them. It is designated $\overline{A B}$.
An endpoint is a point at the end of a segment or ray.
The measure of a line segment is designated $A B$.
$A n$ angle is formed by two rays that share a common endpoint, designated $\angle A B C$ or $\angle D$.
$A$ vertex is the common endpoint of the rays forming the angle.
The measure of an angle is designated $m \angle A B C$ or $m \angle D$.
Congruent (angles, segments, polygons, circles, solids) are identical in size and shape. e.g., $\angle A B C \cong \angle A T C$.

Equal measures of segments or angles are designated $A B=C D$ or $m \angle A B C=m \angle D$. $A$ midpoint of a segment is a point that divides a segment into two congruent segments.
II.

A bisector is a line, segment, or ray that divides a figure into two congruent figures.
$A$ right angle is an angle that measures $90^{\circ}$.
An acute angle is an angle whose measure is between $0^{\circ}$ and $90^{\circ}$.
An obtuse angle is an angle whose measure is between $90^{\circ}$ and $180^{\circ}$.
$A$ straight angle is an angle whose measure is $180^{\circ}$; the rays forming the angle are collinear.
A pair of vertical angles is a pair of non-adjacent angles formed by two intersecting lines. A linear pair of angles consists of two adjacent angles whose sum is $180^{\circ}$.
A pair of complementary angles is a pair of angles whose sum is $90^{\circ}$.
A pair of supplementary angles is a pair of angles whose sum is $180^{\circ}$.
III.

A polygon is a closed figure in a plane formed by connecting line segments endpoint to endpoint.
Consecutive angles in a polygon share one side of the polygon. Consecutive sides in a polygon share one vertex of the polygon.
A convex polygon has all of its diagonals within the polygon.

A concave polygon has at least one diagonal lying outside the polygon.
A diagonal of a polygon is a segment that connects two non-consecutive vertices.
A polygon in which all sides are congruent is an equilateral polygon.
A polygon in which all angles are congruent is an equiangular polygon.
$A$ regular polygon is equilateral and equiangular.
Common polygon names: triangle, quadrilateral, pentagon, hexagon, heptagon, octagon, nine-gon, decagon, dodecagon.
IV.

Perpendicular lines, segments, rays or planes intersect at right angles to each other.
Parallel lines, in the same plane, are equidistant from each other. Parallel planes never intersect.
A right triangle is a triangle that has one right angle.
An acute triangle is a triangle that has three acute angles.
An obtuse triangle is a triangle that has one obtuse angle.
A scalene triangle is a triangle with no congruent sides.
An equilateral triangle is a triangle that has three congruent sides.
An isosceles triangle is a triangle that has at least two congruent sides.
V.

A trapezoid is a quadrilateral with exactly one pair of parallel sides.
$A$ kite is a quadrilateral with two distinct pairs of consecutive congruent sides.
A parallelogram is a quadrilateral with two pairs of parallel sides.
$A$ rhombus is a quadrilateral with four congruent sides.
A rectangle is a quadrilateral with four right angles.
$A$ square is a regular quadrilateral; it has four congruent sides and four right angles.
VI.

A circle is a set of points a given distance (radius) from a given point (center) in the plane.
A diameter is a segment with endpoints on the circle that contains the center of the circle.
An arc of a circle is that part of the circle that consists of two points on the circle and all the points between them. The two points are called endpoints.
A semicircle is an arc of a circle whose endpoints are the endpoints of a diameter.
A chord is a segment whose endpoints lie on the circle.
$A$ tangent is a line that intersects a circle at only one point.
$A$ secant is a line that intersects a circle at two points.

## Terms and Definitions

The following four terms are undefined in the Euclidean axiomatic system.
Describe each term in your own words.
point
line
plane
space
Define and/or draw representations of the following terms using conventional symbols.

Example: ray $\overline{G H}$


A ray is the part of a line that contains point $G$ and all of the points on the same side of point $G$ as point $H$.

| I. | II. |
| :--- | :--- |
| collinear | bisector |
| coplanar | right angle |
| line segment | acute angle |
| endpoint | obtuse angle |
| measure of a line segment | straight angle |
| angle | pair of vertical angles |
| vertex | linear pair of angles |
| measure of an angle | pair of complementary angles |
| congruent | pair of supplementary angles |
| equal measure |  |
| midpoint |  |
|  |  |


| III. |  |
| :--- | :--- |
| polygon |  |
| consecutive angles |  |
| consecutive sides |  |
| convex polygon |  |
| concave polygon |  |
| diagonal | IV. |
| equilateral polygon |  |
| equiangular polygon |  |
| regular polygon |  |
| common polygon names | parallel |
| right triangle |  |
| acute triangle |  |
| obtuse triangle |  |
| V. | equilane triangle |
| trapezoid | isosceles triangle |
| kite |  |
| parallelogram |  |
| rhombus |  |
| rectangle | VI. |
| square | circle |
|  | radius |
|  | center |
|  | diameter |
|  | arc of a circle |
|  | semicircle |
| chord |  |
| tangent |  |

## What is a Translation?

Overview: The properties of translation vectors are determined for 2-dimensional figures in the plane and on the coordinate plane.

## Objective: TExES Mathematics Competencies

II.006.A. The beginning teacher understands the concept of slope as a rate of change and interprets the meaning of slope and intercept in a variety of situations.
III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.014.A. The beginning teacher identifies transformations (i.e., reflections, translations, glide-references, rotations, dilations) and explores their properties.
III.014.D. The beginning teacher applies transformations in the coordinate plane.
III.014.H. The beginning teacher explores the relationship between geometric and algebraic representations of vectors and uses this relationship to solve problems.
V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

## Geometry TEKS

d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.
d.2.B. The student uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons. e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.

## Background: Identify and graph coordinates and slopes in coordinate figures. Recognize parallel lines.

Materials: easel paper, centimeter ruler, colored markers
New Terms: image, isometry, pre-image, translation vector

## Procedures:

The activity has been divided into parts, represented by the different phases of learning in the van Hiele model, which will be explained to participants after they have experienced the activities on translations and reflections. The activity begins with participants sharing prior knowledge about translations, using 1, in their groups. Post a large sheet of easel paper with the heading "Translations."

1. The polygon $A B C D E F G$, transformed into polygon $A^{\prime} B^{\prime} C^{\prime} D^{\prime} E^{\prime} F^{\prime} G^{\prime}$ below, represents a translation. The original figure, $A B C D E F G$, is called the pre-image. The resultant figure, $A^{\prime} B^{\prime} C^{\prime} D^{\prime} E^{\prime} F^{\prime} G^{\prime}$, is called the image. Each point on the image is labeled with the same letter as its corresponding pre-image point, with the addition of a prime mark.


Ask participants to add the new terms pre-image and image to their glossaries.
In your group, discuss the figure and make a list of properties for a translation.
After a few minutes, during whole group discussion, ask a participant to record a list of translation properties supplied by the class. Do not discuss or critique the statements. Explain that, after completing $2-5$, these statements will be critiqued for validity.

## Where are translations seen in the real world?

Some examples of translations can be seen in decorative friezes, rows of apartment units, or hotel rooms facing the same direction, and some tessellations.

Participants work on $2-5$. Move around the room listening and facilitating while participants work.
2. Use a ruler and colored pencil or marker to connect each pre-image point to its corresponding image point. In your group, discuss and record your findings on paper.


The connecting segments are all parallel and have the same magnitude.
3. A translation vector resembles a ray and is used to define the magnitude (by the length of the vector) and the direction (by the direction of the arrow) of the translation. Transform pentagon $O P Q R S$ using the translation vector given. Show the segments connecting pre-image to corresponding image points. Label the image points appropriately.


Remind participants to add the new term translation vector to their glossaries.
4. Draw and number the $x$ - and $y$-axes on the grid below.

Write the coordinates of the vertices of $\Delta U V W$.
Apply the rule $(x, y) \rightarrow(x+3, y-2)$ to the coordinates for $U, V$, and $W$.
Draw the image that results from this transformation.
An example:


|  | Pre-image <br> $(\boldsymbol{x}, \boldsymbol{y})$ | Image |
| :---: | :---: | :---: |
| $(\boldsymbol{x}+\mathbf{3}, \boldsymbol{y}-\mathbf{2})$ |  |  |
| $U$ | $(-2,2)$ | $(1,0)$ |
| $V$ | $(1,-2)$ | $(4,-4)$ |
| $W$ | $(2,0)$ | $(5,-2)$ |

Using a colored pencil or marker, connect corresponding pre-image and image points.
Describe what happened in terms of the transformation rule $(x, y) \rightarrow(x+3, y-2)$.
The figure moved 3 units to the right and 2 units down.
The slope of the translation vector is $-\frac{2}{3}$.
Generalize: Describe the translation for the rule $(x, y) \rightarrow(x+a, y+b)$.
If $a$ and $b$ are positive, each point moves $a$ units to the right and $b$ units up. If $a$ and $b$ are negative, then the point moves in the opposite direction.
The slope of the translation vector is $\frac{b}{a}$.
5. In your group, discuss and record on paper the properties of translations. Be prepared to share during whole class discussion.

After most participants have completed $2-5$, facilitate a whole class discussion on the properties of translations.

Ask a volunteer to record the class-proposed properties of translations on the easel paper. Discuss each item for validity from the list produced at the beginning of the activity. Cross out invalid properties.

The following represent possible responses.

- The pre-image and image are congruent.
- The pre-image and image have exactly the same orientation.
- The pre-image points all move in exactly the same direction and distance.
- Parallel congruent segments connect the pre-image points to the corresponding image points.
- The connecting segments with directional arrows are called translation vectors.
- The translation vector determines the magnitude and direction of movement for each point in the figure.
- If $a$ and $b$ are both positive, for the general coordinate rule $(x, y) \rightarrow(x+a, y+b)$, each point moves $a$ units to the right and $b$ units up. If $a$ and $b$ are negative, then points move a units to the left and $b$ units down.
The slope of the translation vector is $\frac{b}{a}$.
Participants should discuss any additional properties and knowledge emerging from the activity.

Look at 3. It shows a figure with a translation vector.
Does a translation vector have to be attached to the figure?
No, it can lie anywhere. However, if the connecting segments are drawn, they should all be parallel to, and of the same magnitude as, the given translation vector.

## What is an isometry?

An isometry is a transformation that preserves congruence.

## Is translation a type of isometry?

Yes, the pre-image and image are congruent.
Point out that students in secondary school may prefer to use the term slide instead of the term translation. Slide is acceptable at the informal Visual Level. However, following any discussion, when formal terms and symbols have been introduced, only formal terms should be used.

Participants work independently on 6 and 7, applying the properties of translations.
6. Find the coordinate rule for the following translation:


$$
(x, y) \rightarrow(x-2, y+1)
$$

7. Polygon $H^{\prime} E^{\prime} X^{\prime} A^{\prime} G^{\prime} N^{\prime}$ is the image resulting from the translation rule $(x, y) \rightarrow(x+7, y-4)$. Find the coordinates of the pre-image.


$$
\begin{aligned}
& H(-5,6) \\
& E(-4,9) \\
& X(-2,8) \\
& A(0,5) \\
& G(-2,2) \\
& N(-3,5)
\end{aligned}
$$

## What is a Translation?

1. The polygon $A B C D E F G$, transformed into polygon $A^{\prime} B^{\prime} C^{\prime} D^{\prime} E^{\prime} F^{\prime} G^{\prime}$ below, represents a translation. The original figure, $A B C D E F G$, is called the pre-image. The resultant figure, $A^{\prime} B^{\prime} C^{\prime} D^{\prime} E^{\prime} F^{\prime} G^{\prime}$, is called the image. Each point on the image is labeled with the same letter as its corresponding pre-image point, with the addition of a prime mark.


In your group, discuss the figure and make a list of properties for a translation.
2. Use a ruler and colored pencil or marker to connect each pre-image point to its corresponding image point. In your group, discuss and record your findings on paper.
3. A translation vector resembles a ray, and is used to define the magnitude (by the length of the vector) and the direction (by the direction of the arrow) of the translation. Transform pentagon OPQRS using the translation vector given. Show the segments connecting pre-image to corresponding image points. Label the image points appropriately.

4. Draw and number the $x$ - and $y$ - axes on the grid below. Write the coordinates of the vertices of $\Delta U V W$. Apply the rule $(x, y) \rightarrow(x+3, y-$ 2 ) to the coordinates for $U, V$, and $W$. Draw the image that results from this transformation.


|  | Pre-image <br> $(\boldsymbol{x}, \boldsymbol{y})$ | Image <br> $(\boldsymbol{x}+\mathbf{3}, \boldsymbol{y}-\mathbf{2})$ |
| :--- | :---: | :---: |
| $U$ |  |  |
| $V$ |  |  |
| $W$ |  |  |

Using a colored pencil or marker, connect corresponding pre-image and image points. Describe what happened in terms of the transformation rule $(x, y) \rightarrow(x+3, y-2)$.

Generalize: Describe the translation for the rule $(x, y) \rightarrow(x+a, y+b)$.
5. In your group, discuss and record on paper the properties of translations. Be prepared to share during whole class discussion.
6. Find the coordinate rule for the following translation:

7. Polygon $H^{\prime} E^{\prime} X^{\prime} A^{\prime} G^{\prime} N^{\prime}$ is the image resulting from the translation rule $(x, y) \rightarrow(x+7, y-4)$. Find the coordinates of the pre-image.


## 2поitว9[79Я Reflections

Overview: The properties of reflections are determined for 2-dimensional figures in the plane and on the coordinate plane.

Objective: TExES Mathematics Competencies
II.006.A. The beginning teacher understands the concept of slope as a rate of change and interprets the meaning of slope and intercept in a variety of situations.
III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.014.A. The beginning teacher identifies transformations (i.e., reflections, translations, glide-references, rotations, dilations) and explores their properties.
III.014.C. The beginning teacher uses transformations to explore and describe reflectional, rotational, and translational symmetry.
III.014.D. The beginning teacher applies transformations in the coordinate plane.
III.014.H. The beginning teacher explores the relationship between geometric and algebraic representations of vectors and uses this relationship to solve problems.
V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

## Geometry TEKS

d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.
d.2.B. The student uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons.
e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.

Background: Participants should be able to identify slopes and graph coordinates in the coordinate plane and recognize parallel and perpendicular lines.

Materials: easel paper, centimeter ruler, colored markers, patty paper, graph paper

New Terms: line of reflection or line of symmetry

## Procedures:

Post a large sheet of easel paper with the heading "Reflections." Distribute the activity sheet. Ask participants to work only on 1 for about 5 minutes.

1. The figures below represent reflections across the dotted lines. In your group, discuss and record the properties of reflections. Be prepared to share all of your observations at the end of this activity during whole class discussion.

Ask a participant to record reflection properties as provided by the class on the easel paper. Do not critique the statements.

## Where are reflections seen in the real world?

Some examples are: mirrors, adjacent hotel rooms, the word "AMBULANCE" written on the front of an ambulance as seen in a rear-view mirror, works of art (e.g., M.C. Escher).

The mathematical definition of a line of reflection or a line of symmetry is the line over which a figure is reflected resulting in a figure that coincides exactly with the original figure.

Ask participants to add the terms line of reflection and line of symmetry to their glossaries.

Participants work on $2-5$. Walk around the room listening and facilitating while participants work. When most participants have completed these items, conduct a group discussion to bring about consensus.
2. Find the lines of reflection or lines of symmetry for the following figures.

The line of reflection can be obtained by connecting the midpoints of segments drawn between corresponding pre-image and image points. Alternately, connect a preimage point to a non-corresponding image point and vice versa. The intersection point is on the line of reflection. Repeat for a different pair of points. Draw a line, the line of reflection, through the intersection points.

3. Reflect each polygon below across its line of reflection. Label at least one polygon and its image, using prime notation, e.g., $A$ transforms to $A^{\prime}$.

In your group, share and write down observations. Be prepared to share all of your observations at the end of this activity during whole class discussion.

If anyone needs further assistance, remind them to use the property discovered in 2 , or provide patty paper. Have participants trace the figure and the reflection line, fold across the reflection line, and trace the folded figure. They should then copy the result onto the activity sheet. Figure E is difficult. The grid is a hindrance. Give participants adequate time to work on E. Discourage use of patty paper for this item.

4. For each of the figures $\mathrm{A}-\mathrm{D}$ from 2, let one end of the reflection segment represent the origin. Examples are illustrated below. Work with a partner to find the coordinates of each pre-image point and its corresponding image point. Then complete the following:
a. When a figure is reflected across the $y$-axis, $(x, y) \rightarrow(-x, y)$.
b. When a figure is reflected across the $x$-axis, $(x, y) \rightarrow(x,-y)$.
c. When a figure is reflected across the line $y=-x,(x, y) \rightarrow(-y,-x)$.

d. Predict: When a figure is reflected across the line $y=x,(x, y) \rightarrow(y, x)$.

On a coordinate grid, draw a shape, reflect it across the line $y=x$, and verify your prediction.

With your group, share and record on paper your observations.
An example is shown.

5. Describe at least three ways in which the position of the line of reflection, or line of symmetry, can be determined.

- Trace the pre-image and the image. Fold so that the figures coincide. The fold line is the line of reflection.
- Draw a segment from a pre-image point to its corresponding image point. Construct the perpendicular bisector, which is the line of symmetry.
- Draw at least two line segments connecting points to the corresponding image points. Find the midpoints. Draw a line through the midpoints, forming the line of reflection.
- Connect a pre-image point to a non-corresponding image point and vice versa. The intersection point is on the line of reflection. Repeat for a different pair of points. Draw a line, the line of reflection, through the intersection points.

Lead a whole class discussion during which participants share their observations. Discuss the validity of each of the statements provided at the start of the lesson. Ask a participant to record new properties on the poster, and be sure to include all of the following points in the discussion.

- Reflections are congruence transformations.
- Reflections are a type of isometry, because the pre-image and its image are congruent.
- The image and its pre-image are the same distance from the line of reflection but on opposite sides.
- The lines connecting corresponding pre-image to image points are parallel to each other and the line of reflection is their perpendicular bisector.

When these properties are provided, participants should label congruence and perpendicular symbols on their papers. For example:


- If a line is perpendicular to two distinct lines in the plane, then the two lines are parallel to each other.
- If the slope of the line of reflection is $m$, then the slopes of the connecting parallel lines are all $-\frac{1}{m}$.
- Coordinate rules:

When a figure is reflected across the $y$-axis, $(x, y) \rightarrow(-x, y)$.
When a figure is reflected across the $x$-axis, $(x, y) \rightarrow(x,-y)$.
When a figure is reflected across the line $y=-x,(x, y) \rightarrow(-y,-x)$.
When a figure is reflected across the line $y=x,(x, y) \rightarrow(y, x)$.
Point out that the term flip is commonly used in secondary classrooms, but this informal term should be replaced with the formal term reflection.

## 2пoitosItsタ Reflections

1. The figures below represent reflections across the dotted lines. In your group, discuss and record the properties of reflections. Be prepared to share all of your observations at the end of this activity during whole class discussion.

2. Find the lines of reflection, or lines of symmetry, for the following figures.
a.

b.

3. Reflect each polygon below across its line of reflection. Label at least one polygon and its image using prime notation, e.g., $A$ transforms to $A^{\prime}$.

In your group, share and write down observations. Be prepared to share all of your observations at the end of this activity during whole class discussion.

| A | $\square$ |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\cdots$ |  |  |  |  | V | $N$ |  |
|  | $\checkmark$ |  |  |  |  | $\checkmark$ | N |  |
|  |  |  |  |  | B |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  | C |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  | N |  |  |  |  |  |  |  |
|  | $\cdots$ |  |  |  |  |  |  |  |
|  | $\perp$ |  |  |  |  |  |  |  |
|  |  |  |  |  |  | D. |  |  |
|  |  |  |  |  |  | D |  |  |
|  | - |  |  |  |  | $\rightarrow$ |  |  |
|  |  |  |  |  |  | $\cdots$ | 1 |  |
| E |  |  |  |  |  | , | - |  |
|  |  |  |  |  |  |  | $\bigcirc$ |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |

4. For each of the figures $\mathrm{A}-\mathrm{D}$ from 2 , let one end of the reflection segment represent the origin. Work with a partner to find the coordinates of each pre-image point and its corresponding image point. Then complete the following:
a. When a figure is reflected across the $y$-axis, $(x, y) \rightarrow$ $\qquad$
b. When a figure is reflected across the $x$-axis, $(x, y) \rightarrow$ $\qquad$
c. When a figure is reflected across the line $y=-x,(x, y) \rightarrow$
d. Predict:

When a figure is reflected across the line $y=x,(x, y) \rightarrow$
On a coordinate grid, draw a shape, reflect it across the line $y=x$, and verify your prediction.

With your group, share and record on paper your observations.
5. Describe at least three ways in which the position of the line of reflection, or line of symmetry, can be determined.

## Theoretical Framework: The van Hiele Model of Geometric Thought

Overview: Participants view and take notes on the PowerPoint presentation about the van Hiele Model of Geometric Thought, the theoretical framework on which this geometry module is based.

## Objective: TExES Mathematics Competencies

VI.020.A. The beginning teacher applies research-based theories of learning mathematics to plan appropriate instructional activities for all students.
VI.020.B. The beginning teacher understands how students differ in their approaches to learning mathematics.
VI.020.C. The beginning teacher uses students' prior mathematical knowledge to build conceptual links to new knowledge and plans instruction that builds on students' strengths and addresses students' needs.
VI.020.E. The beginning teacher understands how to provide instruction along a continuum from concrete to abstract.
VI.020.F. The beginning teacher understands a variety of instructional strategies and tasks that promote students' abilities to do the mathematics described in the TEKS.
VI.020.G. The beginning teacher understands how to create a learning environment that provides all students, including English Language Learners, with opportunities to develop and improve mathematical skills and procedures.
VI.020.H. The beginning teacher understands a variety of questioning strategies to encourage mathematical discourse and to help students analyze and evaluate their mathematical thinking.

## Background:

Materials: PowerPoint presentation: van Hiele Model of Geometric Thought or transparencies of PowerPoint slides

## New Terms:

## Procedures:

This activity describes the theoretical framework known as the van Hiele model for geometric thought as it relates to the Geometry Module. Awareness of the model helps teachers understand how geometry should be taught and identify reasons for possible students' lack of success in learning high school geometry.

Students first learn to recognize whole shapes and then to analyze the relevant properties of the shapes. Later they see relationships between shapes and make simple deductions. Curriculum development and instruction must consider this hierarchy because although
learning can occur at several levels simultaneously, the learning of more complex concepts and strategies requires a firm foundation of prior skills.

The van Hiele model underscores the importance of the Learning Principle in NCTM's Principles and Standards for School Mathematics (2000), p. 11, which states that "students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge." Students learn more mathematics and learn it better when they can take control of their learning by defining their goals and monitoring their progress. Effective learners recognize the importance of reflecting on their thinking and learning from their mistakes.

Distribute the activity sheet to participants to help them focus on the important ideas from the PowerPoint presentation "The van Hiele Model of Geometric Thought" as they take notes. They will need additional paper for note-taking. Show the PowerPoint presentation. Use the following notes pages to elaborate on the content of each slide.
$\left.\begin{array}{c|c|}\hline \text { Slide } \\ 1\end{array} \begin{array}{c}\text { The van Hiele Model } \\ \text { of Geometric } \\ \text { Thought }\end{array}\right]$

The van Hiele model of geometric thought outlines the hierarchy of levels through which students progress as they develop geometric ideas. The model clarifies many of the shortcomings in traditional instruction and offers ways to improve it. Pierre van Hiele and his wife, Dina van Hiele-Geldof, focused on getting students to the appropriate level to be successful in high school geometry.


Language is the basis for understanding and communicating. Before we can find out what a student knows we must establish a common language and vocabulary.

| Slide | When is it appropriate <br> to ask for a definition? |
| :--- | :--- |
| A definition of a concept is only <br> possible if one knows, to some <br> extent, the thing that is to be <br> defined. |  |



| Slide 5 | Levels of Thinking in Geometry <br> - Visual Level <br> - Descriptive Level <br> - Relational Level <br> - Deductive Level <br> - Rigor |
| :---: | :---: |

Only through the use of a common language can we develop the network of structures and relations which help produce good definitions.

The development of geometric ideas progresses through a hierarchy of levels. The research of Pierre van Hiele and his wife, Dina van Hiele-Geldof, clearly shows that students first learn to recognize whole shapes, then to analyze the properties of a shape. Later they see relationships between the shapes and make simple deductions. Only after these levels have been attained can they create deductive proofs.


The hierarchy for learning geometry described by the van Hieles parallels Piaget's stages of cognitive development. One should note that the van Hiele model is based on instruction, whereas Piaget's model is not.

The van Hiele model supports Vygotsky's notion of the "zone of proximal development" which is the "distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (Vygotsky, 1978, p. 85-86).

Language at the Visual Level serves to make possible communication for the whole group about the structures that students observe. Formal or conventional vocabulary is used to describe the figures. Any misconceptions identified may be clarified by the use of appropriate language. The language of the next level (e.g., congruence) will not be understood by students who are at the Visual Level.

In this example, the term rotation is introduced.


The term reflection is introduced. Students may note that a mirror might produce a reflection.

| Slide 12 | Descriptive Level Characteristics <br> The student <br> - recognizes and describes a shape (e.g., parallelogram) in terms of its properties. <br> - discovers properties experimentally by observing, measuring, drawing and modeling. <br> - uses formal language and symbols. <br> - does NOT use sufficient definitions. Lists many properties. <br> - does NOT see a need for proof of generalizations discovered empirically (inductively). |
| :---: | :---: |

The language of the Descriptive Level includes words relating to properties within a given figure (e.g., parallel, perpendicular, and congruent). Various properties can be used to describe or define a figure, but this is usually a very uneconomical list of properties. A concise definition, using a sufficient number of properties rather than an exhaustive list, is not possible at this level. Students functioning at the Visual Level are not able to understand the properties of the Descriptive Level even with the help of pictures.


The isosceles triangles, with congruent vertices at the point or center of rotation, become an accepted property of rotations. Students at the Descriptive Level recognize a rotation by these properties.

The congruent, parallel translation vectors become the defining property behind a translation. The symbols used to describe the vectors (e.g., parallel and congruent), both on the figure and in written geometric language, become a part of the formal language of the Descriptive Level.

Slide


The line of reflection bisects the parallel segments that connect corresponding points on the pre-image and image. This property describes and defines a reflection. The congruency symbols, the perpendicular symbol, and the corresponding symbols used in written descriptions become part of the language of the Descriptive Level.

| Slide $17$ | Relational Level Characteristics <br> The student <br> can define a figure using minimum (sufficient) sets of properties. gives informal arguments, and discovers new properties by deduction. <br> - follows and can supply parts of a deductive argument. <br> does NOT grasp the meaning of an axiomatic system, or see the interrelationships between networks of theorems. |
| :---: | :---: |

Slide

18 | Relational Level |
| :--- |
| Example |

Slide
19

## Deductive Level

My experience as a teacher of geometry convinces me that all too often, students have not yet achieved this level of informal deduction. Consequently, they are not successful in their study of the kind of geometry that Euclid created, which involves formal deduction.

Pierre van Hiele

Slide
Deductive Level Characteristics

The student

- recognizes and flexibly uses the components of an axiomatic system (undefined terms, definitions, postulates, theorems).
- creates, compares, contrasts different proofs.
- does NOT compare axiomatic systems.


| Slide | Phases of the |
| :---: | :--- |
|  | Instructional Cycle |
|  | - Information |
|  | - Guided orientation |
|  | Explicitation |
|  | - Integration |
|  |  |
|  |  |




Slide
26

## Guided Orientation Phase

- The activities guide the student toward the relationships of the next level.
- The relations belonging to the context are discovered and discussed.


The teacher provides instructional activities in which students explore and discuss the concept, preferably in small groups, and come to a consensus about the concept within their groups. The teacher's role is to facilitate, provide hints, and ask scaffolding questions rather than to provide answers. Students should construct their own knowledge from their own thinking rather than rely on the teacher for direct information.

After the class has completed the guided orientation activity, the teacher leads a whole class discussion. Each group shares its findings, and through discussion misconceptions are re-conceptualized. Even if groups share the same conclusions, much can be gained when students hear explanations in different words or from slightly different viewpoints. The van Hiele model emphasizes that this step is often short-changed in mathematics classrooms at the expense of student understanding and learning.


The knowledge gained by completing the instructional cycle now forms the basis for the Information Phase of the next level of thinking.
Using the newly found properties of the rhombus, students should be able to work backwards to sketch these rhombi.
Sometimes students arrive at the wrong conclusions. During the discussion, these misconceptions can be corrected, especially if other students, rather than the teacher, are able to explain to their peers.

After the whole class discussion, the teacher provides independent practice using the newly discovered relationships. This can be in the form of homework problems or extended investigations.

Slide

33 \begin{tabular}{l}
Integration Phase <br>

| Summarize and memorize the properties of |
| :--- |
| a rhombus. | <br>

\hline
\end{tabular}

Slide
34

## What we do and what we do not do...

- It is customary to illustrate newly introduced technical language with a few examples.
- If the examples are deficient, the technical language will be deficient.
- We often neglect the importance of the third stage, explicitation. Discussion helps clear out misconceptions and cements understanding.

| Slide 35 | What we do and what we do not do... <br> - Sometimes we attempt to inform by explanation, but this is useless. Students explanation. <br> - The teacher must give guidance to the process of learning, allowing students to them find their way in the field of thinking |
| :---: | :---: |

The traditional form of teaching, by modeling and explanation, is time efficient but not effective. Students learn best through personal exploration and active thinking. By answering students' questions, we rob students of the opportunity to develop good thinking habits. It is best to guide by asking probing or scaffolding questions, making suggestions, and waiting.
At the next level of thinking, students must know these properties for immediate recall.

The use of the appropriate language and its symbols for each level is critical if students are to progress to higher levels. If the language is mismatched for the level at which the student functions, then he/she will not be able to progress, even if the teacher attempts to use visual structures for explanation.

Slide

Instructional Considerations

- Visual to Descriptive Level
- Language is introduced to describe figures that are observed.
- Gradually the language develops to form the background to the new structure.
- Language is standardized to facilitate
communication about observed properties.
- It is possible to see congruent figures, but it is useless to ask why they are congruent.

| Slide | Instructional <br> Considerations <br> - Descriptive to Relational Level <br> - Causal, logical or other relations become <br> part of the language. <br> - Explanation rather than description is <br> possible. <br> - Able to construct a figure from its known <br> properties but not able to give a proof. |
| :---: | :--- |

Language is critical in moving students through the hierarchy.

Slide | Instructional |
| :--- |
| Considerations |
| - Relational to Deductive Level |
| - Reasons about logical relations between |
| theorems in geometry. |
| - To describe the reasoning to someone who does |
| not "speak" this language is futile. |
| - At the Deductive Level it is possible to arrange |
| arguments in order so that each statement, |
| except the first one, is the outcome of the |
| previous statements. |

| Slide | Instructional <br> Considerations |
| :---: | :--- |
| • Rigor |  |
| - Compares axiomatic systems. |  |
|  | -Explores the nature of logical laws. |
| "Logica/Mathematical/Thinking" |  |


| $\begin{gathered} \text { Slide } \\ 40 \end{gathered}$ | Consequences <br> - Many textbooks are witten with only the <br> - The integration phase often coincides with <br> the objective of the eearning. <br> - Many teachers switch to, or even begin <br> Leaching. <br> - Many teachers do not realize that their |
| :---: | :---: |


| Slide |
| :--- |
| 41 | | Children whose geometric thinking you <br> nurture carefully will be better able to <br> successfully study the kind of <br> mathematics that Euclid created. |
| :--- |
| Pierre van Hiele |

The following table provides more detailed descriptors for what a student should be able to do at each van Hiele level. These are for your reference or for anyone requiring more information. The descriptors were adapted from Fuys, Geddes, \& Tischler (1988).

| Level | Descriptors |
| :---: | :---: |
| Visual Level | 1. Identifies instances of a shape by its appearance as a whole in <br> - a simple drawing, diagram or set of cut-outs. <br> - different positions. <br> - a shape or other more complex configurations. <br> 2. Constructs (using craft or commercial materials), draws, or copies a shape. <br> 3. Names or labels shapes and other geometric configurations and uses standard and/or nonstandard names and labels appropriately. <br> 4. Compares and sorts shapes on the basis of their appearance as a whole. <br> 5. Verbally describes shapes by their appearance as a whole. <br> 6. Solves routine problems by operating on shapes rather than by using properties which apply in general (e.g., by overlaying, measuring, counting). <br> 7. Identifies parts of a figure but <br> - does NOT analyze a figure in terms of its components. <br> - does NOT think of properties as characterizing a class of figures. <br> - does NOT make generalizations about shapes or use related language. |

\(\left.$$
\begin{array}{|c|c|}\hline & \begin{array}{l}\text { 1. Identifies and tests relationships among components of figures (e.g., } \\
\text { congruence of opposite sides of a parallelogram; congruence of } \\
\text { angles in a tiling pattern). }\end{array} \\
& \begin{array}{l}\text { 2. Recalls and uses appropriate vocabulary for components and } \\
\text { relationships (e.g., corresponding angles are congruent, diagonals } \\
\text { bisect each other). }\end{array}
$$ <br>
3. Compares two shapes according to relationships among their <br>
components. Sorts shapes in different ways according to certain <br>

properties. This may include sorting figures by class or non-class.\end{array}\right\}\)| 4. Interprets and uses verbal description of a figure in terms of its |
| :--- |
| properties and uses this description to draw/construct the figure. |
| Interprets verbal or symbolic statements of rules and applies them. |
| 2. Discovers properties of specific figures empirically and generalizes |
| properties for that class of figures. |


| Relational Level | 1. Identifies different sets of properties that characterize a class of figures and tests that these are sufficient. Identifies minimum sets of properties that can characterize a figure. Formulates and uses a definition for a class of figures. <br> 2. Gives informal arguments (using diagrams, cutout shapes that are folded, or other materials). <br> - Having drawn a conclusion from given information, justifies the conclusion using logical relationships. <br> - Orders classes of shapes. <br> - Orders two properties. <br> - Discovers new properties by deduction. <br> - Interrelates several properties in a family tree. <br> 3. Gives informal deductive arguments. <br> - Follows a deductive argument and can supply parts of an argument. <br> - Gives a summary or variation of a deductive argument. <br> - Gives deductive arguments on own. <br> 4. Gives more than one explanation to prove something and justifies these explanations by using family trees. <br> 5. Informally recognizes difference between a statement and its converse. <br> 6. Identifies and uses strategies or insightful reasoning to solve problems. <br> 7. Recognizes the role of deductive argument and approaches problems in a deductive manner but <br> - does NOT grasp the meaning of deduction in an axiomatic sense (e.g., does NOT see the need for definitions and basic assumptions). <br> - does NOT formally distinguish between a statement and its converse. <br> - does NOT yet establish interrelationships between networks of theorems. |
| :---: | :---: |

\(\left.$$
\begin{array}{|c|c|}\hline & \begin{array}{c}\text { 1. Recognizes the need for undefined terms, definitions, and basic } \\
\text { assumptions (postulates). }\end{array}
$$ <br>
2. Recognizes characteristics of a formal definition (e.g., necessary and <br>
sufficient conditions) and equivalence of definitions. <br>
2. Proves in axiomatic setting relationships that were explained <br>
informally in the relational level. <br>
L. Proves relationships between a theorem and related statements (e.g., <br>

converse, inverse, contrapositive).\end{array}\right\}\)| 5. Establishes interrelationships among networks of theorems. |
| :--- |
| 6. Compares and contrasts different proofs of theorems. |
| 7. Examines effects of changing an initial definition or postulate in a |
| logical sequence. |

## The van Hiele Model of Geometric Thought

Use this sheet and additional paper if needed to take notes from the van Hiele PowerPoint presentation.

Levels of Geometric Understanding
Student characteristics:
Visual Level
Descriptive Level
Relational Level
Deductive Level
Rigor
Phases in the Instructional Cycle:
Knowledge
Guided Orientation
Explicitation
Free Orientation
Integration
Important key instructional considerations:

## Rotations

Overview: In this activity, participants explore the properties of rotations by rotating a variety of figures about different rotation points using different angles of rotation.

## Objective: TExES Mathematics Competencies

III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.014.A. The beginning teacher identifies transformations (i.e., reflections, translations, glide-reflections, rotations, dilations) and explores their properties.
III.014.C. The beginning teacher uses transformations to explore and describe reflectional, rotational, and translational symmetry. III.014.D. The beginning teacher applies transformations in the coordinate plane.
V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.
VI.020.A. The beginning teacher applies research-based theories of learning mathematics to plan appropriate instructional activities for all students.

## Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
d.2.A. The student uses one-and two-dimensional coordinate systems to represent points, lines, line segments, and figures.
e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.

Background: Participants must be able to recognize and apply different angle measurements and to know and apply clockwise and counterclockwise movements. An understanding of translations and reflections is helpful.

New Terms: identity rotation, point of rotation
Materials: centimeter ruler, patty paper, protractor, colored pencils, centimeter grid paper, small colored dot, transparency sheets (1 per group of 4), two overhead pens of different colors (for each group)

## Procedures:

Begin by asking a participant to demonstrate a rotation about a given point, represented by the colored dot on the floor. There are two possible ways to rotate about this point. The participant may stand on the point and turn in a circular motion or he/she may stand off the point and walk around it in a circular path. Depending on the method chosen by the participant, ask for a second volunteer to demonstrate the other possible rotation.

## What is a rotation?

For now, accept all definitions from participants. The purpose is just to get participants thinking about rotations. Possible responses may include a spin on a point or a turn about a point.

## Where are rotations used in the real world?

Possible answers are: architecture, art, kaleidoscopes, tri-mirrors in dressing rooms at clothing stores, Ferris wheels, and an ice skater spinning about a point.


Model how to draw a rotation using patty paper. Ask participants to draw a smiling circular face on patty paper. This is the pre-image. A small arrow pointing upwards can be drawn on the paper to indicate the orientation of the pre-image.

Model or discuss how to rotate the figure $90^{\circ}$ counterclockwise about one of the eyes. The point of rotation is the vertex of the angle of rotation; it is a fixed point. Place the pencil on one of the dots in the middle of an eye to indicate the point of rotation. The angle of rotation can be directed counter-clockwise (having a positive measure) or clockwise (having a negative measure.


Place a second piece of patty paper on top of the smiling face and trace the face. With the pencil on the point of rotation (the dot within one of the eyes), turn the top piece of patty paper $90^{\circ}$ left (counter-clockwise) while keeping the bottom piece of patty paper still. The resulting figure is the image of a $90^{\circ}$ counter-clockwise rotation. Trace the pre-image on the top sheet of patty paper, and draw the angle of rotation with the point of rotation as the vertex of the angle. Using a colored pencil, draw in other angles of rotation by connecting points on the pre-image to the point of rotation and then to the corresponding points on the image.


Next, model how to rotate the face from a point not on the figure. Place a point off of the smiling face and use it as the point of rotation. Ask participants to rotate the smiling face about a point not on the smiling face $-90^{\circ}$, which is a $90^{\circ}$ clockwise rotation. Use a colored pencil to draw the angles of rotation.

Participants work 1-4 to identify the properties of rotations. When participants have completed $1-4$, conduct a whole group discussion on the properties of rotations.

Possible properties participants may identify are:

- Rotations preserve congruence of the two shapes.
- The distance (side length), angle measure, perpendicularity and parallelism are all preserved.
- Rotations are isometries because congruence of size and shape are preserved.
- $180^{\circ}$ clockwise and $180^{\circ}$ counter-clockwise rotations are identical to each other.

1. Rotate $\triangle A B C 180^{\circ}$ counter-clockwise about point $D$. Label the corresponding vertices. Write the coordinates in the table below in order to find the rule for a $180^{\circ}$ counterclockwise rotation.

| Pre-image |  | Image |
| :--- | :--- | :--- |
| $A(-4,0)$ |  | $A^{\prime}(4,0)$ |
| $B(-3,2)$ |  | $B^{\prime}(3,-2)$ |
| $C(4,0)$ |  | $C^{\prime}(-4,0)$ |
| Rule $:(x, y)$ | $\rightarrow$ | $(-x,-y)$ |



Angles of rotation: $\angle B D B, \angle C D C^{\prime \prime}$ and $\angle A D A^{\prime}$
2. Rotate $\triangle A B C 180^{\circ}$ clockwise about point $D$. Write the coordinates of the vertices of the pre-image and the image in the table below. Use the table to find the rule for an $180^{\circ}$ clockwise rotation.

| Pre-image |  | Image |
| :--- | :--- | :--- |
| $A(0,1)$ |  | $A^{\prime}(0,-1)$ |
| $B(3,2)$ |  | $B^{\prime}(-3,-2)$ |
| $C(-2,-3)$ |  | $C^{\prime}(2,3)$ |
| Rule: $(x, y)$ | $\rightarrow$ | $(-x,-y)$ |

Angles of rotation: $\angle B D B^{\prime}, \angle C D C^{\prime}$ and $\angle A D A^{\prime}$


In order to fully understand the geometry of rotations, carefully draw the isosceles triangles for at least one of the rotations in 2, 3 or 4 . Connect a pre-image point to a corresponding image point, in addition to the associated angle of rotation. The vertex of this isosceles triangle is at the point of rotation. Repeat for a second pair of points on the same figure. See the answers below for a full development of this property of rotations, which is required for 6 and 7 .
3. Rotate quadrilateral $P Q R S 90^{\circ}$ counter-clockwise about point $X$.
a) Write the coordinates of the vertices of the pre-image and the image in the table below. Use the table to find the rule for a $90^{\circ}$ counter-clockwise rotation.
b) Draw isosceles triangles $Q X Q^{\prime}$ and $S X S^{\prime}$. Shade each triangle in a different color. Mark congruent segments.
c) Write the measures of $\angle Q X Q^{\prime}$ and $\angle S X S^{\prime}$.


In this solution, two of the four possible isosceles triangles have been drawn. Triangle $Q X Q^{\prime}$ and $\triangle S X S^{\prime}$ are isosceles triangles. The vertices of both triangles meet at the point of rotation, $X$.
4. Rotate quadrilateral $A B C D 90^{\circ}$ clockwise about point $C$.
a) Write the coordinates of the vertices of the pre-image and the image in the table below. Use the table to find the rule for a $90^{\circ}$ clockwise rotation.
b) Draw isosceles triangles $A C A^{\prime}$ and $D C D^{\prime}$. Shade each triangle in a different color. Mark congruent segments.
c) Write the measures of $\angle A C A^{\prime}$ and $\angle D C D^{\prime}$.


Participants work $5-9$. Construct the rotations using a ruler and protractor without the use of patty paper. In 8 and 9 , participants determine the point of rotation and the angle of rotation.

The role of the instructor is to walk around the room, listen and facilitate. Try not to directly answer participants' questions, but rather provide minimal prompts and/or ask questions to help participants clarify their thinking. When most of the participants have completed $5-9$, give each group an overhead transparency sheet and two different colored overhead pens. Each group prepares an answer to one of 5-9 for presentation to the class.
5. Rotate the pentagon $A B E D C 45^{\circ}$ counter-clockwise about point $G$, draw the angles of rotation and connect the corresponding vertices to form isosceles triangles.
The figure below represents the correct rotation. The isosceles triangles shown are three of five possibilities.

6. Rotate the quadrilateral $H I J K-70^{\circ}$ about point $L$, draw the angles of rotation and connect the corresponding vertices to form isosceles triangles.
The figure below represents the correct rotation.
The isosceles triangles shown are two of four possibilities.

7. Describe the effect of rotations of magnitude $0^{\circ}$ and $360^{\circ}$ on a figure.

The image coincides with its pre-image. Each pre-image point is mapped on the corresponding image point. A rotation of magnitude of $0^{\circ}$ or $360^{\circ}$ is the identity rotation.

Find the point of rotation and the angle of rotation in 8 and 9.
Draw a segment connecting a pre-image point to its corresponding image point. Find the midpoint. Using the corner of a sheet of paper, draw a line through the midpoint, perpendicular to the segment. Repeat for another pair of points. The two perpendicular lines pass through the vertex of the isosceles triangles. The vertex is the point of rotation. Measure the vertex angles of the isosceles triangles to find the angle of rotation.
8. Point $F$ is the point of rotation because all of the isosceles triangles share a vertex at $F$. The angle of rotation is $45^{\circ}$ clockwise.


$$
\begin{aligned}
& m \angle A F A^{\prime}=45^{\circ} \\
& m \angle B F B^{\prime}=45^{\circ} \\
& m \angle C F C^{\prime}=45^{\circ} \\
& m \angle D F D^{\prime}=45^{\circ} \\
& m \angle E F E^{\prime}=45^{\circ}
\end{aligned}
$$

9. Point $V$ is the point of rotation, the vertex of the isosceles triangles $\triangle U V U^{\prime}, \Delta T V T^{\prime}$, and $\Delta S V S^{\prime}$.

The angle of rotation is $100^{\circ}$.

$$
\begin{aligned}
& m \angle U V U^{\prime}=100^{\circ} \\
& m \angle T V T^{\prime}=100^{\circ} \\
& m \angle S V S^{\prime}=100^{\circ}
\end{aligned}
$$



At the end of the activity, summarize by asking participants to discuss which van Hiele levels are represented.

The Visual Level is represented in the walking demonstration as participants are asked to experience rotation holistically. Success with the rest of the activity indicates participants are at the Descriptive Level as they are developing and applying properties of transformations.

## Rotations

For each of the following figures construct the given rotations. Draw each angle of rotation in a different color.

1. Rotate $\triangle A B C 180^{\circ}$ counter-clockwise about point $D$. Label the corresponding vertices. Write the coordinates in the table below in order to find the rule for a $180^{\circ}$ counter-clockwise rotation.


|  | Pre-image | Image |
| :---: | :--- | :--- |
| $A$ |  |  |
| $B$ |  |  |
| $C$ |  |  |

2. Rotate $\triangle A B C 180^{\circ}$ clockwise about point $D$. Write the coordinates of the vertices of the pre-image and the image in the table below. Use the table to find the rule for an $180^{\circ}$ clockwise rotation.


|  | Pre-image | Image |
| :---: | :--- | :--- |
| $A$ |  |  |
| $B$ |  |  |
| $C$ |  |  |

Rule: $(x, y) \rightarrow$
3. Rotate quadrilateral $P Q R S 90^{\circ}$ counter-clockwise about point $X$.
a) Write the coordinates of the vertices of the pre-image and the image in the table below. Use the table to find the rule for a $90^{\circ}$ counterclockwise rotation.
b) Draw isosceles triangles $Q X Q^{\prime}$ and $S X S^{\prime}$. Shade each triangle in a different color. Mark congruent segments.
c) Write the measures of $\angle Q X Q^{\prime}$ and $\angle S X S^{\prime}$.


|  | Pre-image | Image |
| :---: | :--- | :--- |
| $P$ |  |  |
| $Q$ |  |  |
| $R$ |  |  |
| $S$ |  |  |

Rule: $(x, y) \rightarrow$
4. Rotate quadrilateral $A B C D 90^{\circ}$ clockwise about point $C$.
a) Write the coordinates of the vertices of the pre-image and the image in the table below. Use the table to find the rule for a $90^{\circ}$ clockwise rotation.
b) Draw isosceles triangles $A C A^{\prime}$ and $D C D^{\prime}$. Shade each triangle in a different color. Mark congruent segments.
c) Write the measures of $\angle A C A^{\prime}$ and $\angle D C D^{\prime}$.


|  | Pre-image | Image |
| :--- | :--- | :--- |
| $A$ |  |  |
| $B$ |  |  |
| $C$ |  |  |
| $D$ |  |  |

5. Rotate the pentagon $A B E D C 45^{\circ}$ counter-clockwise about point $G$, draw the angles of rotation and connect the corresponding vertices to form isosceles triangles.


- $G$

6. Rotate the quadrilateral $H I J K-70^{\circ}$ about point $L$, draw the angles of rotation and connect the corresponding vertices to form isosceles triangles.

7. Describe the effect of rotations of magnitude $0^{\circ}$ and $360^{\circ}$ on a figure.

Find the points of rotation and the angles of rotation in 8 and 9 .
8.

9.


## Composite Transformations

Overview: Participants reflect figures over two lines of reflection to produce a composite transformation.

Objective: TExES Mathematics Competencies
III.014.A. The beginning teacher identifies transformations (i.e., reflections, translations, glide-reflections, rotations, dilations) and explores their properties.
III.014.D. The beginning teacher applies transformations in the coordinate plane.
V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

## Geometry TEKS

c.2. The student uses properties of transformations and their compositions to make connections between mathematics and the real world in applications such as tessellations or fractals.
d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.

Background: Participants need to have a general knowledge of transformations.
Materials: protractor, centimeter ruler, transparency sheets (1 per group of 4), transparency pens in at least two different colors

New Terms: composite transformation or glide-reflection

## Procedures

This activity may be used as a homework assignment. If time permits, have participants complete the activity in class. Participants work individually, and then compare answers with their group members.

A composite transformation is created when one transformation follows another transformation. For example, a reflection is followed by a reflection, or a reflection is followed by a translation.

1. Create a composite reflection by reflecting the flag pre-image over line $l$ and then reflecting the image over line $m$.
2. Measure the acute angle created by the intersection of lines $l$ and $m$. For each of the two reflections, measure the angles created between a pre-image point, the point of intersection of $l$ and $m$, and the corresponding image point. What conclusion can be drawn?

When a figure is reflected about two intersecting lines, the resulting image is the same as that from a rotation. The measure of the angle of rotation is twice the measure of the angle between the intersecting lines. The point of intersection between the two lines is the point of rotation.
m


$$
\begin{aligned}
& m \angle I B J=129^{\circ} \\
& m \angle K B L=64.5^{\circ} \\
& m \angle I B J=2 m \angle K B L
\end{aligned}
$$

3. Create a composite reflection over parallel lines. Reflect the flag figure about line $m$ and then reflect the image over line $l$. Write a conclusion about a composite reflection over two parallel lines.


When a figure is reflected over parallel lines, the resulting image is the same as that created from translating the original figure. The magnitude of the translation vector is twice the distance between the parallel reflection lines. The direction of the translation is along the line perpendicular to the parallel lines.
4. Translate the flag figure with the rule $(x, y) \rightarrow(x+3, y-5)$. Then reflect the image across line $m$. The composite transformation is called a glide-reflection. Describe the properties of a glide-reflection.


A glide-reflection resembles footprints on either side of a reflection line. The translation and reflection can be performed in any order resulting in the same image.
5. Use the grid below to create "footprints in the sand". Draw a foot at one end of the grid below and create a set of footprints using the properties of glide reflections. A possible answer is shown.


Ask participants to add the new terms composite transformation and glide-reflection to their glossaries.

Success with this activity indicates that participants are at the Descriptive Level with respect to congruence transformations as a whole, because they apply properties of each transformation in each of the steps and then identify a transformation based on its properties.

## Composite Transformations

A composite transformation is created when one transformation follows another transformation. For example, a reflection is followed by a reflection, or a reflection is followed by a translation.

1. Create a composite reflection by reflecting the flag pre-image over line $l$ and then reflecting the image over line $m$.

2. Measure the acute angle created by the intersection of lines $l$ and $m$. For each of the two reflections, measure the angles created between a preimage point, the point of intersection of $l$ and $m$, and the corresponding image point. What conclusion can be drawn?
3. Create a composite reflection over parallel lines. Reflect the flag figure about line $m$ and then reflect the image over line $l$. Write a conclusion about a composite reflection over two parallel lines.

4. Translate the flag figure with the rule $(x, y) \rightarrow(x+3, y-5)$. Then reflect the image across line $m$. The composite transformation is called a glidereflection. Describe the properties of a glide-reflection.

5. Use the grid below to create "footprints in the sand". Draw a foot at one end of the grid below and create a set of footprints using the properties of glide reflections.


## Tessellations

Overview: Participants explore the properties of triangles and parallel lines by tessellating a scalene triangle.

## Objective: TExES Mathematics Competencies

III.012.C. The beginning teacher applies the properties of parallel and perpendicular lines to solve problems.
III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
III.014.B. The beginning teacher uses the properties of transformations and their compositions to solve problems.
III.014.C. The beginning teacher uses transformations to explore and describe reflectional, rotational, and translational symmetry.

## Geometry TEKS

b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
c.2. The student uses properties of transformations and their compositions to make connections between mathematics and the real world in applications such as tessellations or fractals.
e.2.A. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties of parallel and perpendicular lines.
e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.

Background: Participants must understand translations and rotations.
Materials: centimeter ruler, patty paper, protractor, colored pencils, unlined 8.5 in. by 11 in . paper, 3 in . by 5 in . index card cut in half, 11 in . by 17 in . sheet of paper, colored markers, masking tape, easel paper

New Terms: tessellation

## Procedures:

Look around the room for a tessellation. Be ready to explain what you know about tessellations. Possible answers are the tiles on the ceiling or floor. Explain that a tessellation is a pattern made up of one or more shapes, completely tiling the surface, with no gaps and no overlaps. The original figure is called the fundamental region.

According to the dictionary, tessellate means to form or arrange small squares in a checkered or mosaic pattern. The word tessellate is derived from the Ionic version of the Greek word tesseres, which, in English, means four. The first tessellations were made from square tiles (Alejandre, 2004).

What properties are required in order for a figure to tessellate?
At each vertex, the angle measures must add to $360^{\circ}$; if vertices meet at a point on a line, the angle measures must add to $180^{\circ}$.

Participants create a tessellation using an acute scalene triangle or an obtuse scalene triangle but not a right scalene triangle. The triangle will be the fundamental region of the tessellation. Draw the triangle on a one-half index card. Cut out the triangle to use as a template. Shade each of the three angles of the triangle in a different color.

Create the tessellation using only rotations and translations, and not reflections. The tessellation may not have gaps or overlaps.

Two illustrations of non-acceptable placements follow.


One possible tessellation is shown:


When most participants have completed their tessellations, ask each group to discuss the triangle properties and the parallel line and angle properties which emerge from the tessellation. Each group records their properties on large easel paper with diagrams as needed. Then each group displays their poster on the wall.

Participants walk around the room as a group noting relationships they did not discover. Bring the participants back for a whole group discussion about tessellations. Using the overhead tessellation, individual group members summarize and explain one property their group discovered.

Make sure that the following relationships arise.
Triangle properties:

- The sum of the angles of a triangle is $180^{\circ}$. (Three angles of different colors that line up along a line are the same as the three colors for the three angles of the triangle.)
- The exterior angle of a triangle (made up of two colors) is equal to the sum of the measures of the two nonadjacent interior angles (the same two colors).

Parallel line properties created by transversal lines:

- Alternate interior angles are congruent (same color).
- Alternate exterior angles are congruent (same color).
- Interior angles on the same side of the transversal are supplementary (three colors altogether).
- Exterior angles on the same side of the transversal are supplementary (three colors altogether).
- Corresponding angles are congruent (same color).

Other relationships:

- Vertical angles are congruent (same color).
- The sum of the measures of the angles of a quadrilateral is $360^{\circ}$ (two sets of the three colors).
- The sum of the measures of the angles of a convex polygon of $n$ sides is $(n-2) 180^{\circ}$ ( $n-2$ sets of the three colors).
- The sum of the exterior angles of a polygon is $360^{\circ}$ (two sets of the three colors).


## A regular polygon has congruent sides and angles. Which regular polygons tessellate the plane?

The triangle, square, and hexagon are the only regular polygons that tessellate the plane. The measure of each angle of the equilateral triangle is $60^{\circ}$, and six $60^{\circ}$-angles tessellate around a point. The measure of each angle of a square is $90^{\circ}$, and four $90^{\circ}$-angles tessellate around a point. The measure of each angle of a hexagon is $120^{\circ}$, and three $120^{\circ}$ angles tessellate around a point.

Success with this activity indicates that participants are performing at the Descriptive Level since they apply properties of translations and rotation, and determine properties among parallel lines and triangles.

Tessellation


## Do You See What I See?

Overview: After exploring two-dimensional transformations, participants explore transformations of three-dimensional objects by building, sketching the solids top, front, and side views then comparing the volume and surface area of the reflected solid to the original solid.

Objective: TExES Mathematics Competencies
III.011.B. The beginning teacher applies formulas for perimeter, area, surface area, and volume of geometric figures and shapes (e.g., polygons, pyramids, prisms, cylinders).
III.013.D. The beginning teacher computes the perimeter, area, and volume of figures and shapes created by subdividing and combining other figures and shapes (e.g., arc length, area of sectors). III.013.F. The beginning teacher uses top, front, side, and corner views of three-dimensional shapes to create complete representations and solve problems.
III.014.C. The beginning teacher uses transformations to explore and describe reflectional, rotational, and translational symmetry.

## Geometry TEKS

d.1.C. The student uses top, front, side, and corner views of threedimensional objects to create accurate and complete representations and solve problems.
e.1.D. The student finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations. e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.

Background: Participants should be able to identify and describe reflections, rotations, and translations in a coordinate plane.

Materials: linking cubes, plastic mirrors, a small object such as a color tile for each participant

New Terms: surface area, volume

## Procedures:

Participants should begin by constructing the following three-dimensional solid with linking cubes. The instructor should model the construction of the solid shown below and guide participants through their construction.


Front

Using linking cubes, construct a three-dimensional solid.

1. Sketch the solid on the isometric grid paper below.

2. Sketch the top, front, and right views of the solid.

| TOP View |  |  |  |  |  | FRONT View |  |  |  |  |  | RIGHT View |  |  |  |  |
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3. Define the unit of volume as a single cube. Find the volume of this solid. Justify your answer.
The volume of the solid is 14 cubes, since the bottom layer has 9 cubes, the middle layer has 4 cubes, and the top layer has 1 cube.
4. Define total surface area as the total number of unit squares on the outer surface of the solid, including the base that rests on the table, regardless of its orientation. Define a unit of area as a single square. Find the total surface area of the solid. Justify your answer.
The surface area of the solid is 42 squares. The top layer has 5 squares, the middle layer has 11 squares, and the bottom layer has 26 squares.
5. Place a mirror parallel to one view of the solid. Using the mirror as a plane of reflection, predict what the reflected image of the solid would look like. Using the mirror as a plane of reflection, build the reflection of the solid.


Front


Sketch the reflected solid on the isometric grid paper below.

6. Sketch the top, front, and right views of the reflected solid.

| TOP View |  |  |  |  |  | FRONT View |  |  |  |  | RIGHT View |  |  |  |  |  |
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7. What is the volume of the reflected solid? Justify your answer.

The volume of the solid is 14 cubes, since the bottom layer has 9 cubes, the middle layer has 4 cubes, and the top layer has 1 cube.
8. How does the volume of the reflected solid compare to the volume of the original solid?
The volume is the same as that of the original solid.
9. What is the surface area of the reflected solid? Justify your answer.

The surface area of the solid is 42 squares. The top layer has 5 squares, the middle layer has 11 squares, and the bottom layer has 26 squares.
10. How does the surface area of the reflected solid compare to the surface area of the original solid?
The surface area is the same as the original solid.
11. Restore the solid to its original orientation. Place a marker about two inches from one corner of the solid. Using the marker as a point of rotation, rotate the solid $90^{\circ}$ counter-clockwise in the plane of the table upon which you constructed the solid.


Sketch the rotated solid on the isometric grid paper below.

12. Sketch the top, front, and right views of the rotated solid.

13. How does the volume and surface area of the rotated solid compare to the volume and surface area of the original solid? Why?
The volume and surface area are the same as the original solid, because rotation is an isometry; rotation preserves congruence.

Remind participants to add the new terms total surface area and volume to their glossaries.

Participants begin this activity working at the Visual Level as they build and draw a copy of a three-dimensional solid in 1 . The transition between the Visual Level and the Descriptive Level occurs in 2-3 as participants analyze the appearance of threedimensional top, front, and right view and translate the observations to a two-dimensional representation. Participants are beginning to work at the Descriptive Level in 4-13 as they determine volume and surface area using known properties of the solid, reflect and rotate the solid, and compare the volume and surface area of the two figures. Beginning at the Visual Level, participants develop visual perception of solids. To be successful at this activity, the participant must be fluently working at the Visual Level of van Hiele's model.

## Do You See What I See?

Using linking cubes, construct a three-dimensional solid.

1. Sketch the solid on the isometric grid paper below.

2. Sketch the top, front, and right views of the solid.

| TOP View |  |  |  |  |  | FRONT View |  |  |  |  |  | RIGHT View |  |  |  |  |
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3. Define the unit of volume as a single cube. Find the volume of this solid. Justify your answer.
4. Define total surface area as the total number of unit squares on the outer surface of the solid, including the base that rests on the table, regardless of its orientation. Define a unit of area as a single square. Find the total surface area of this solid. Justify your answer.
5. Place a mirror parallel to one view of the solid. Using the mirror as a plane of reflection, predict what the reflected image of the solid would look like. Using the mirror as a plane of reflection, build the reflection of the solid. Sketch the reflected solid on the isometric grid paper below.
6. Sketch the top, front, and right views of the reflected solid.

| TOP View |  |  |  |  |  | FRONT View |  |  |  |  | RIGHT View |  |  |  |  |  |
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7. What is the volume of the reflected solid? Justify your answer.
8. How does the volume of the reflected solid compare to the volume of the original solid?
9. What is the surface area of the reflected solid? Justify your answer.
10. How does the surface area of the reflected solid compare to the surface area of the original solid?
11. Restore the solid to its original orientation. Place a marker about two inches from one corner of the solid. Using the marker as a point of rotation, rotate the solid $90^{\circ}$ counter-clockwise in the plane of the table upon which you constructed the solid. Sketch the rotated solid on the isometric grid paper below.
12. Sketch the top, front, and right views of the rotated solid.

| TOP View |  |  |  |  |  |  | FRONT View |  |  |  |  |  |  | RIGHT View |  |  |  |  |  |
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13. How does the volume and surface area of the rotated solid compare to the volume and surface area of the original solid? Why?

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## Unit 2 - Triangles

## Equilateral Triangles

Overview: In this activity participants discover properties of equilateral triangles using properties of symmetry.

## Objective: TExES Mathematics Competencies

III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.012.E. The beginning teacher describes and justifies geometric constructions made using compass and straightedge, reflection devices, and other appropriate technologies.
III.013.A. The beginning teacher analyzes the properties of polygons and their components.
III.013.B. The beginning teacher analyzes the properties of circles and the lines that intersect them.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).

## Geometry TEKS

b.1.A. The student develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems.
b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.4. The student uses a variety of representations to describe geometric relationships and solve problems.
c.3. The student identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-6090) and triangles whose sides are Pythagorean triples.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.
e.2.C. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of circles and the lines that intersect them.

Background: Participants need a basic understanding of reflections, lines of reflection, lines of symmetry, and properties of similar figures.

Materials:
patty paper, straightedge compass, easel paper, colored markers
New Terms: altitude, central angle, circumscribed circle, concentric circle, inscribed angle, inscribed circle, line of reflection or line of symmetry, median, perpendicular bisector

## Procedures:

Distribute the activity sheet, easel paper, and markers to participants. Have them work for about 45 minutes, answering the questions individually on their own paper. Allow discussion with others when needed. Participants record their responses to 12 on a sheet of easel paper. During whole group discussion, ask one participant from each group to summarize the properties determined by his/her group. Ask other participants to add any properties omitted by earlier groups. Be careful to list all of the properties that appear below, including the connections to special right triangles, similar figures and circles from 10 and 11.

Complete 1-12 to explore the properties of equilateral triangles.

1. Carefully construct a large equilateral triangle on patty paper using a straightedge and compass. Label $\Delta S T V$.

2. List properties of equilateral triangles and mark the triangle to indicate the identified properties. Explain how you know these properties from the constructed triangle.
Participants may write that equilateral triangles have equal side lengths and equal angle measures. These properties can be verified by folding the angles on top of each other.
3. Fold, draw or construct the lines of symmetry/reflection on the patty paper triangle. Label the intersection of the lines of symmetry $P$.
4. Label the intersection of the line of symmetry from vertex $S$ to $\overline{T V}$ as $A$, the intersection of the line of symmetry from vertex $V$ to $\overline{T S}$ as $B$, and the intersection of the line of symmetry from vertex $T$ to $\overline{S V}$ as $C$.

5. Describe the relationship between the lines of symmetry and the vertex angles of the triangle. Mark these properties on the triangle. Label the measures of the angles created by the lines of symmetry. Based on these properties, what is another name for the lines of symmetry? The lines of symmetry can be called angle bisectors because the segments bisect the vertex angles, forming two $30^{\circ}$ angles.

6. Describe the relationship between the lines of symmetry and the sides of the triangle. Mark the relationships on the triangle. The lines of symmetry bisect the sides of the triangle.
$\overline{T B} \cong \overline{S B} ; \overline{S C} \cong \overline{V C} ; \overline{T A} \cong \overline{V A}$.
The lines of symmetry are perpendicular to the sides of the
 triangle. $\overparen{V B} \perp \overline{T S} ; \quad \overparen{S A} \perp \overline{T V} ; \quad \overleftrightarrow{T C} \perp \overline{S V}$.
7. Using the information now marked on the triangle, what are three other names which can be used to describe the segments lying on the lines of symmetry?
Because the three lines of symmetry bisect the sides of the triangle at right angles, the segments lying on the lines of symmetry are also medians, altitudes and perpendicular bisectors. A median of a triangle is a segment connecting a vertex to the midpoint of the opposite side of the triangle. An altitude is a segment from a vertex, perpendicular to the opposite side. A perpendicular bisector is a segment that bisects another segment at a right angle.
8. Use a ruler or patty paper to compare the lengths of the medians.

The medians, which lie on the lines of symmetry, are congruent.
9. Use a ruler or patty paper to compare the lengths of the longer and shorter segments that make up a median, for example, $\overline{T P}$ and $\overline{P C}$. Find the ratio of their lengths. Why is the ratio the same for all three medians?
The ratio of the longer to the shorter segment is $2: 1$. For example, $T P=2 \cdot P C$. This holds true for all of the medians because the symmetry properties ensure congruence for all of the longer segments and all of the shorter segments.
10. Compare the six smaller right triangles and the six larger right triangles formed by the medians. Describe the relationship between the larger right triangles and the smaller right triangles.
All of the right triangles have angle measures of $30^{\circ}, 60^{\circ}$, and $90^{\circ}$. We call these $30^{\circ}-60^{\circ}-90^{\circ}$ triangles. The larger and smaller right triangles are similar. Similar shapes
 have congruent angles, and proportional side lengths. In the large $30^{\circ}-60^{\circ}-90^{\circ}$ triangle, the length of the hypotenuse, a side of the equilateral triangle, is twice the length of the short side, formed by the bisection of a side of the equilateral triangle. The relationship must exist for both the small and large triangles. This explains the 2:1 ratio for the medians.
11. Using a compass, draw a circle inscribed in $\triangle T S V$, radius $\overline{P B}$. Draw a second circle circumscribed about $\triangle T S V$, radius $\overline{P T}$. Record the properties of an inscribed circle and a circumscribed circle for an equilateral triangle.

- The length of the radius of the circumscribed circle is twice the length of the radius of the inscribed circle in an equilateral triangle.

- The two circles have the same center; they are concentric circles.
- $\angle S P V \cong \angle T P V \cong \angle T P S$. These are central angles for the circumscribed circle, because their vertices are located at the center of the circle, and their sides are radii of the circle. $\angle S T V, \angle T S V$, and $\angle T V S$ are inscribed angles, intercepted by the same chords, $\overline{S V}, \overline{T V}$, and $\overline{T S}$, as the central angles, in that order. Note that the measures of these central angles are $120^{\circ}$, while the measures of the corresponding inscribed angles are $60^{\circ}$. The measure of the central angle is equal to twice the measure of the inscribed angle intercepted by the same chord.

12. On a sheet of easel paper, construct and label an equilateral triangle with the lines of symmetry, and the inscribed and circumscribed circles. List all of the properties of equilateral triangles.

- Equilateral triangles have congruent sides and congruent angles.
- Equilateral triangles have three lines of reflectional symmetry.
- The vertex angles measure $60^{\circ}$.
- The three lines of symmetry bisect the vertex angles.
- The three segments that lie on the lines of symmetry are angle bisectors, altitudes, medians and perpendicular bisectors.
- The lines of symmetry divide the triangle into six larger and six smaller $30^{\circ}-60^{\circ}$ $90^{\circ}$ triangles.
- The larger and smaller right triangles are similar.
- Within all $30^{\circ}-60^{\circ}-90^{\circ}$ triangles, the length of the hypotenuse is twice the length of the short leg.
- The inscribed and circumscribed circles are concentric circles centered at the intersection of the lines of symmetry.
- The radius of the circumscribed circle is twice as long as the radius of the inscribed circle.
- The measure of a central angle is two times the measure of the inscribed angle intercepted by the same chord.

At the end of the activity, facilitate a whole group discussion about the van Hiele levels used throughout the activity on equilateral triangles. Participants making observations in $1-4$ are performing on the Visual Level, using quick observation. With $5-12$, the questions move participants to the Descriptive Level because an exhaustive list of properties is developed by further observation and measurement. Although the relationships between different properties are explored in these problems, the questions revolve around one triangle, the equilateral triangle.

## Equilateral Triangles

Complete $1-12$ to explore the properties of equilateral triangles.

1. Carefully construct a large equilateral triangle on patty paper using ruler and compass. Label $\Delta S T V$.
2. List properties of equilateral triangles and mark the triangle to indicate the identified properties. Explain how you know these properties from the constructed triangle.
3. Fold, draw or construct the lines of symmetry/reflection on the patty paper triangle. Label the intersection of the lines of symmetry $P$.
4. Label the intersection of the line of symmetry from vertex $S$ to $\overline{T V}$ as $A$, the intersection of the line of symmetry from vertex $V$ to $\overline{T S}$ as $B$, and the intersection of the line of symmetry from vertex $T$ to $\overline{S V}$ as $C$.
5. Describe the relationship between the lines of symmetry and the vertex angles of the triangle. Mark these properties on the triangle. Label the measures of the angles created by the lines of symmetry. Based on these properties, what is another name for the lines of symmetry?
6. Describe the relationship between the lines of symmetry and the sides of the triangle. Mark the relationships on the triangle.
7. Using the information now marked on the triangle, what are three other names which can be used to describe the segments lying on the lines of symmetry?
8. Use a ruler or patty paper to compare the lengths of the medians.
9. Use a ruler or patty paper to compare the lengths of the longer and shorter segments that make up a median, for example, $\overline{T P}$ and $\overline{P C}$. Find the ratio of their lengths. Why is the ratio the same for all three medians?
10. Compare the six smaller right triangles and the six larger right triangles formed by the medians. Describe the relationship between the larger right triangles and the smaller right triangles.
11. Using a compass, draw a circle inscribed in $\triangle T S V$ with radius $\overline{P B}$. Draw a second circle circumscribed about $\Delta T S V$ with radius $\overline{P T}$. Record the properties of an inscribed circle and a circumscribed circle for an equilateral triangle.
12. On a sheet of easel paper, construct and label an equilateral triangle with the lines of symmetry and the inscribed and circumscribed circles. List all of the properties of equilateral triangles.

## Two Congruent Angles

Overview: Participants construct right, acute, and isosceles obtuse triangles using the radii of congruent circles to determine properties of isosceles triangles. An extension introduces and extends properties of circle segments and angles.

## Objective: TExES Mathematics Competencies

III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.013.A. The beginning teacher analyzes the properties of polygons and their components.
III.013.B. The beginning teacher analyzes the properties of circles and the lines that intersect them.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles). V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.
VI.020.A. The beginning teacher applies research-based theories of learning mathematics to plan appropriate instructional activities for all students.

## Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships. b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.
e.2.C. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of circles and the lines that intersect them.

Background: Participants must understand rotational symmetry.
Materials: patty paper, straightedge, protractor, compass, easel paper, colored markers

## New Terms:

## Procedures:

Distribute the activity sheet. Post a large sheet of easel paper headed "Isosceles Triangles."

1. The figure represents an isosceles triangle. In your group discuss and write down the properties of isosceles triangles that can be observed in the given figure. Be prepared to share these in whole class discussion.


Allow about 5 minutes for group discussion on 1. Ask a participant to record properties on the easel paper, while others share their group's properties. The properties will be critiqued at the end of the activity.
2. You will need three sheets of patty paper.

- Construct three congruent circles, one on each sheet of patty paper.
- On the first sheet, label the center $O_{1}$. Draw two radii forming a right angle at the center of the circle. Draw the chord connecting the endpoints of the radii, $I_{l}$ and $S_{l}$, forming a isosceles right triangle, $\Delta I_{l} S_{l} O_{l}$.
- On the second sheet, label the center $O_{2}$. Draw two radii forming an acute angle at the center of the circle. Connect the endpoints of the radii to complete the triangle, an isosceles acute triangle, $\Delta I_{2} \mathrm{~S}_{2} \mathrm{O}_{2}$.
- On the third sheet, label the center $O_{3}$. Draw two radii forming an obtuse angle at the center of the circle. Complete the triangle, an isosceles obtuse triangle, $\Delta I_{3} S_{3} O_{3}$.

- Find and crease the line(s) of symmetry associated with each triangle. Based on the properties of reflection, determine additional properties of isosceles triangles. In your group, share your findings. Make a group list of properties shared by all isosceles triangles and separate lists for properties unique to isosceles right, isosceles acute and isosceles obtuse triangles.

Allow participants 15-20 minutes to construct the figures and discuss the properties. Then ask participants to provide additional properties to be added to the easel paper poster. After all properties have been posted, critique the properties provided at the beginning of the lesson. Participants justify why the properties are true based on the symmetry properties within these triangles. If participants are unable to generate properties, try to prompt with probing questions rather than providing the list of properties.

Possible property statements (justifications in parentheses):

- Any isosceles triangle has one line of symmetry bisecting the vertex angle and the base. The median to the base lies on the line of symmetry and connects the vertex angle to the midpoint of the base.
- Isosceles triangles have congruent legs (symmetry property).
- Isosceles triangles have congruent base angles (symmetry property).
- The median to the base is also the altitude to the base and the perpendicular bisector of the base (the symmetry line); it creates right angles where it intersects and bisects the base, as well as bisecting the vertex angle (symmetry property).

Isosceles right triangles:


- The median to the base creates two smaller isosceles right triangles (The base angles of the original triangle each measure $45^{\circ}$. The right vertex angle is bisected to form two $45^{\circ}$ angles.)
- The median is congruent to the two segments formed by the median on the base. (These are the legs of the smaller isosceles right triangles created by the line of symmetry.)
- The midpoint of the base is the circumcenter of the triangle. (The three segments radiating from this point to the vertices of the triangle are all congruent, forming radii of the circumscribed circle.)
- The length of the base is twice the length of the altitude (the median).

Isosceles acute triangle:

- The length of the base is shorter than two times the length of the altitude.

Isosceles obtuse triangle:

- The length of the base is longer than two times the length of the altitude.


## Extension investigations:

3. The bases of the isosceles triangles are chords of congruent circles. The vertex angles are the central angles subtended by the chords. The lengths of the altitudes to the bases are the distances from the chords to the centers of the circles. Examine your figures with this in mind, and in your group make conjectures relating chord length to the distance from the center of the circle.
The shorter the length of the chord, the greater is its distance from the center of the circle. The longer the length of the chord, the shorter is its distance is from the center of the circle. As the measure of the central angle increases, the length of the chord increases.

If participants are unable to see the chord and distance to center relationships, encourage them to overlay the centers of the three circles and the lines of symmetry.


## Can the chord continue to grow longer indefinitely?

No, the longest chord is the diameter, which passes through the center of the circle. Its central angle measures $180^{\circ}$.
4. Place points, labeled $N_{1}, N_{2}$, or $N_{3}$, respectively on each circle, so that major arcs $I_{1} N_{1} S_{1}, I_{2} N_{2} S_{2}$, and $I_{3} N_{3} S_{3}$ are formed. Draw inscribed angles $\angle I_{1} N_{1} S_{1}$, $\angle I_{2} N_{2} S_{2}$, and $\angle I_{3} N_{3} S_{3}$. A conjecture regarding the central angle and the inscribed
angle was made in the activity on equilateral triangles. Use each of your circles to verify the relationship of the central angle to the inscribed angle that intercepts the same arcs.


The measure of the central angle is equal to twice the measure of the inscribed angle intercepted by the same arc.
Participants may offer equivalent statements.
At this time an inductive approach, using a protractor or patty paper, is appropriate for the van Hiele Descriptive Level of understanding. Participants may be familiar with the theorem that the measure of the central angle is twice the measure of the inscribed angle intercepted on the same arc. It is important that they understand that the deductive process can usually be followed by those proficient at the Relational Level. This activity guides development of the Descriptive Level, which expects properties to emerge out of inductive approaches. Discuss the different van Hiele levels presented in this activity. In 1 and 2 participants are performing at the Visual Level. Recognition of the figures is all that is needed to do the constructions.

In 2 and 3, the work on the properties moves participants to the Descriptive Level, but approaches the Relational Level in 4. In 2 and 3, the circle is used to describe the properties of the isosceles triangle. In 4, properties of circles are connected to those of isosceles triangles.

## Two Congruent Angles

1. The figure represents an isosceles triangle. In your group, discuss and write down the properties of isosceles triangles that can be observed in the given figure. Be prepared to share these in whole class discussion.

2. You will need three sheets of patty paper.

- Construct three congruent circles, one on each sheet of patty paper.
- On the first sheet, label the center $O_{l}$. Draw two radii forming a right angle at the center of the circle. Draw the chord connecting the endpoints of the radii, $I_{l}$ and $S_{l}$, forming a isosceles right triangle, $\Delta I_{l} S_{l} O_{l}$.
- On the second sheet, label the center $O_{2}$. Draw two radii forming an acute angle at the center of the circle. Connect the endpoints of the radii to complete the triangle, an isosceles acute triangle, $\Delta I_{2} \mathrm{~S}_{2} \mathrm{O}_{2}$.
- On the third sheet, label the center $O_{3}$. Draw two radii forming an obtuse angle at the center of the circle. Complete the triangle, an isosceles obtuse triangle, $\Delta I_{3} S_{3} O_{3}$.
- Find and crease the line(s) of symmetry associated with each triangle. Based on the properties of reflection, determine additional properties of isosceles triangles. In your group, share your findings. Make a group list of properties shared by all isosceles triangles and separate lists for properties unique to isosceles right, isosceles acute and isosceles obtuse triangles.

Extension investigations:
3. The bases of the isosceles triangles are chords of congruent circles. The vertex angles are the central angles subtended by the chords. The lengths of the altitudes to the bases are the distances from the chords to the centers of the circles. Examine your figures with this in mind, and in your group make conjectures relating chord length and distance from the center of the circle.
4. Place points, labeled $N_{1}, N_{2}$, or $N_{3}$, respectively on each circle, so that major $\operatorname{arcs} I_{1} N_{1} S_{1}, I_{2} N_{2} S_{2}$, and $I_{3} N_{3} S_{3}$ are formed. Draw inscribed angles $\angle I_{1} N_{1} S_{1}, \angle I_{2} N_{2} S_{2}$, and $\angle I_{3} N_{3} S_{3}$. A conjecture regarding the central angle and the inscribed angle was made in the activity on equilateral triangles. Use each of your circles to verify the relationship of the central angle to the inscribed angle that intercepts the same arcs.

## Scalene Triangles

Overview: Participants determine properties that apply to all scalene triangles.

> Objective: TExES Mathematics Competencies III.012.B. The beginning teacher uses properties of points, lines, planes, angles, lengths, and distances to solve problems. III.012.E. The beginning teacher describes and justifies geometric constructions made using compass and straightedge, reflection devices, and other appropriate technologies. III.013.A. The beginning teacher analyzes the properties of polygons and their components. III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).

## Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.

Background: Participants need a knowledge of the properties of isosceles and equilateral triangles.

Materials: patty paper, centimeter ruler, compass, protractor

## New Terms:

## Procedures:

Distribute the activity sheet. Participants answer 1-3 regarding the properties of scalene triangles and triangle inequalities. The following is taken from Discovering Geometry: an Investigative Approach, $3^{\text {rd }}$ Edition, ©2003, with permission from Key Curriculum Press.

These triangles are scalene triangles.


These triangles are not scalene triangles.


1. What is a scalene triangle?

A scalene triangle is a triangle with no congruent sides and no congruent angles.
2. Name three types of scalene triangles.

Scalene triangles can be scalene right, scalene acute or scalene obtuse.
3. Using the three scalene triangles above, explore and summarize the properties of scalene triangles for symmetry.
No properties result from symmetry, because scalene triangles have no symmetry.
4. Investigate the measures of the angles in triangles and the lengths of the sides opposite those angles. Draw a scalene triangle on your paper. Measure each angle using a protractor. Using a ruler, measure the length of each side in centimeters, rounding to the nearest tenth of a centimeter. Label the measures of the angles and side lengths. What relationship exists among the measures of the angles and the side lengths of the triangle?
The largest angle is opposite the longest side; the smallest angle is opposite the shortest side; the remaining angle is opposite the remaining side.

The following is taken from Discovering Geometry: an Investigative Approach, $3^{\text {rd }}$ Edition, ©2003, p 214, with permission from Key Curriculum Press.
5. Investigate geometric inequalities using either patty paper or a compass and straightedge. Complete each construction and compare your results with group members.

Construct triangles with each set of segments as sides.
a)


These segments do not create a triangle.
b)


These segments create a triangle.
c) Describe the relationship among the lengths of the segments needed to create a triangle.
The sum of the lengths of any two sides of a triangle must be greater than the length of the third side.
6. In the figure, $B C=B D$.

a. Why is $A D>C A$ ?

In $\triangle A C D, \overline{C A}$ is opposite $\angle 3$ and $\overline{A D}$ is opposite $\angle A C D$. $\angle 2 \cong \angle 3$ ( $\triangle B C D$ is an isosceles triangle.) $m \angle A C D=m \angle 1+m \angle 2$ (The whole must be equal to the sum of its parts.)

By substitution, $m \angle A C D=m \angle 1+m \angle 3$.
Therefore $m \angle A C D>m \angle 3$, and $A D>C A$ (The larger side is opposite the larger angle; the smaller side is opposite the smaller angle.)
b. Why is $A B+B C>C A$ ?
$A D=A B+B D$.
By substitution, $A D=A B+B C$, since $B C=B D$.
Since $A D>C A$, from part $a, A B+B C>C A$.
At the end of the activity, discuss the van Hiele levels represented in the activity. In the first three problems, participants perform at the Visual Level, where the language of the concept is clarified. In 3-5 participants determine and describe properties of triangles, at the Descriptive Level. In 6 the steps of the proof are provided, allowing participants to approach the Relational Level. If the proof had been created with only the given information, then participants would have been performing on the Deductive Level.

## Scalene Triangles

These triangles are scalene triangles.


These triangles are not scalene triangles.


1. What is a scalene triangle?
2. Name three types of scalene triangles.
3. Using the three scalene triangles above, explore and summarize the properties of scalene triangles for symmetry.
4. Investigate the measures of the angles in triangles and the lengths of the sides opposite those angles. Draw a scalene triangle on your paper. Measure each angle using a protractor. Using a ruler, measure the length of each side in centimeters, rounding to the nearest tenth of a centimeter. Label the measures of the angles and side lengths. What relationship exists among the measures of the angles and the side lengths of the triangle?
5. Investigate geometric inequalities using either patty paper or a compass and straightedge. Complete each construction and compare your results with group members.

Construct triangles with each set of segments as sides.
a)

b)

c) Describe the relationship among the lengths of the segments needed to create a triangle.
6. In the figure, $B C=B D$.

a. Why is $A D>C A$ ?
b. Why is $A B+B C>C A$ ?

## The Meeting Place

Overview: Participants construct the circumcenter, incenter, centroid and orthocenter in congruent triangles and then determine which points of concurrency lie on the Euler Line.

## Objective: TExES Mathematics Competencies

II.006.B. The beginning teacher writes equations of lines given various characteristics (e.g., two points, a point and slope, slope and y-intercept). II.006.G. The beginning teacher models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology.
III.012.B. The beginning teacher uses properties of points, lines, angles, lengths, and distances to solve problems.
III.013.B. The beginning teacher analyzes the properties of circles and the lines that intersect them.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two-and threedimensional figures and shapes (e.g., relationships of sides, angles).
III.014.E. The beginning teacher applies concepts and properties of slope, midpoint, parallelism, perpendicularity, and distance to explore properties of geometric figures and solve problems in the coordinate plane.

## Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships. b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.4. The student uses a variety of representations to describe geometric relationships and solve problems.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
d.2.B. The student uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.
e.2.C. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of circles and the liens that intersect them.

Background: Participants must be able to write a linear equation given two points, or a point and a slope, and to solve systems of linear equations algebraically.

Materials: patty paper, centimeter ruler, compass, calculator
New Terms: centroid, circumcenter, Euler line, incenter, orthocenter

## Procedures:

## Part 1: Constructions with Patty Paper

Participants use patty paper to construct a perpendicular bisector, a line perpendicular to a given line from a point not on the line, and an angle bisector.

The directions for the three constructions are given on the activity sheet. Complete each construction separately. Discuss findings as each construction is completed.

The following constructions will be used in Part 2.

1. Perpendicular Bisector of a Segment

Draw a segment on a sheet of patty paper. Label the endpoints $A$ and $B$. By folding, construct the perpendicular bisector of the segment. Mark a point anywhere on the perpendicular bisector. Label the point $C$. Draw $\overline{A C}$ and $\overline{B C}$. Measure $A C$ and $B C$. Write a conjecture relating $C$ to $\overline{A B}$.
All points on a perpendicular bisector are equidistant from the endpoints of the bisected segment.

2. Perpendicular to a Segment Through a Point Not on the Segment Draw a segment on patty paper. Mark a point, $E$, not on the segment. By folding, construct a perpendicular line to the segment, passing through $E$.
3. Angle Bisector

Draw an angle, labeled $\angle F G H$ on patty paper. By folding, construct the angle bisector. Mark a point, $I$, on the angle bisector. Using another sheet of patty paper for a right angle tool, or folding, construct the perpendicular lines from $I$ to the sides of the angle, $\overrightarrow{G F}$ and $\overrightarrow{G H}$. Write a conjecture relating the distance from a point on the angle bisector to the sides of the angle.


The points on an angle bisector are equidistant from the rays forming the angle.

## Part 2: Points of Concurrency

4. On a new sheet of patty paper, draw a scalene triangle covering at least half of the sheet. Trace the triangle onto four more sheets, so that you have five congruent triangles. Label each $\triangle X Y Z$.
5. On the first triangle, construct the perpendicular bisectors of the three sides by folding. Label the point of concurrency $C$. Measure and compare the distances from $C$ to the vertices of the triangle. Explain why these distances must be equal.
Since $C$ lies on the perpendicular bisector of $\overline{Y Z}$, from $1, C Y=C Z$.
Since C lies on the perpendicular bisector of $\overline{X Y}$, from $1, C X=C Y$.
By substitution $C X=C Y=C Z$.


Draw the circumscribed circle, centered at $C$, through the vertices of the triangle. Point $C$ is called the circumcenter. Label the sheet "Circumcenter - Perpendicular Bisectors."
6. On the second triangle, construct the three angle bisectors by folding. Label the point of concurrency $I$. Measure and compare the perpendicular distances from $I$ to the sides of the triangle. Mark the points where the perpendiculars from $I$ intersect the sides of the triangle. Explain why these distances must be equal.
From 2, I is equidistant from $\overline{X Z}$ and $\overline{X Y}$, since $\overline{X I}$ bisects $\angle Z X Y$. Similarly, I is equidistant from $\overline{X Z}$ and $\overline{Y Z}$. Therefore, I is equidistant from all three sides of the triangle.


Draw the inscribed circle, centered at $I$, through the marked points on the sides of the triangle. Point $I$ is called the incenter. Label the sheet "Incenter - Angle Bisectors."
7. On the third triangle, by folding, pinch the midpoints of each of the sides of the triangle. With a ruler, draw the three medians, the segments connecting each vertex to the midpoint of the opposite sides. Label the point of concurrency $M$. Measure and determine the ratio into which $M$ divides each median. Label the sheet "Centroid Medians."

An example is shown below.
$Z M=4.42 \mathrm{~cm} \quad M T=2.21 \mathrm{~cm}$
$\frac{Z M}{M T}=2.00$
$X M=3.92 \mathrm{~cm} \quad M R=1.96 \mathrm{~cm}$
$\frac{X M}{M R}=2.00$

8. On the fourth triangle, by folding, construct the three altitudes (the perpendicular segments from each vertex to the opposite side). If you have an obtuse triangle you will need to extend two of the sides outside the triangle. Label the point of concurrency $O$. Label the sheet "Orthocenter - Altitudes."

9. Place the fifth triangle directly on top of the first triangle, so that the vertices coincide exactly. Mark and label point $C$. Repeat with the other three points of concurrency, $I$,
$M$, and $O$. Which three of the four points are collinear (lie on the same line)? Draw in the line connecting these three points. Label the sheet "Euler Line." The Euler line is named after Leonhard Euler, a Swiss mathematician (1708-1783), who proved that the three points of concurrency are collinear.


## Part 3: Algebraic Connections

The points of concurrency can be found algebraically using knowledge of slope, $y$-intercept, perpendicular lines, the writing of linear equations, and the solving of systems of linear equations.

In the following problems, algebraically find the circumcenter, centroid, orthocenter, and the equation of the Euler line.

Each problem requires the same algebraic work. Discuss methods for finding each point.

## How do we find the circumcenter?

The circumcenter is the intersection of the perpendicular bisectors of the sides of the triangle. Find the equations of two of the perpendicular bisectors, and then find their intersection point. To find each equation, use the midpoint of one side of the triangle and the slope of the perpendicular to that side.

## How do we find the centroid?

The centroid is the intersection of the medians. Find the equations of two of the medians, and then find their intersection point. To find each equation, use the midpoint of a side and the vertex opposite that side.

## How do we find the orthocenter?

The orthocenter is the intersection of the altitudes. Find the equations of two of the altitudes, and then find their intersection point. To find each equation, use the slope of the perpendicular to a side and the vertex opposite that side.

## How do we find the equation of the Euler Line?

Use two of the points of concurrency to find the equation, and then check by substituting the coordinates for the third point.

Each group should be given a different problem (10-12). The work for each problem may be divided among the members of the group. Then the whole group should come together to share the results and complete the problem. Participants should graph their triangles to help visualize the figures, and also to check the algebraic work. This section may be assigned for homework.

The solution for 10 follows. Answers only are given for 11 and 12.
10. Triangle $A B C$ has vertices $A(0,8), B(-3,-1)$, and $C(5,-2)$.

Circumcenter: $\left(\frac{3}{2}, \frac{5}{2}\right)$
Centroid: $\left(\frac{2}{3}, \frac{5}{3}\right)$
Orthocenter: $(-1,0)$
Euler Line: $y=x+1$.
11. Triangle DEF has vertices $\mathrm{D}(-2,0)$, $\mathrm{E}(-6,-2)$, and $\mathrm{F}(3,-5)$.

Circumcenter: $(-2,-5)$
Centroid: $\left(-\frac{5}{3},-\frac{7}{3}\right)$
Orthocenter: $(-1,3)$
Euler Line: $y=8 x+11$.
12. Triangle GHJ has vertices $\mathrm{G}(3,7), \mathrm{H}(-1,-1)$, and $\mathrm{J}(5,-4)$.

Circumcenter: $\left(4, \frac{3}{2}\right)$
Centroid: $\left(\frac{7}{3}, \frac{2}{3}\right)$
Orthocenter: $(-1,-1)$
Euler Line: $y=\frac{1}{2} x-\frac{1}{2}$
A possible solution for 10 :
$A(0,8), B(-3,-1), C(5,-2)$.

## Circumcenter:

$$
\begin{aligned}
& \text { Midpoint of } \overline{B C} \text { : } \\
& \left(\frac{-3+5}{2}, \frac{-1-2}{2}\right)=\left(1, \frac{-3}{2}\right) \\
& \text { Slope of } \overline{B C}: \frac{-1-(-2)}{-3-5}=-\frac{1}{8} .
\end{aligned}
$$

$$
\begin{aligned}
& \text { Midpoint of } \overline{A B}: \\
& \left(\frac{0-3}{2}, \frac{8-1}{2}\right)=\left(\frac{-3}{2}, \frac{7}{2}\right) . \\
& \text { Slope of } \overline{A B}: \frac{8-(-1)}{0-(-3)}=\frac{9}{3}=3 .
\end{aligned}
$$

Perpendicular slope $=8$.
Equation: $y=8 x+b$.
To find $b$, substitute $\left(1,-\frac{3}{2}\right)$ in the above equation.

$$
\begin{aligned}
& -\frac{3}{2}=8+b \\
& b=-\frac{3}{2}-8=-\frac{19}{2}
\end{aligned}
$$

The equation of the perpendicular bisector of $\overline{B C}$ is $y=8 x-\frac{19}{2}$.

Perpendicular slope $=-\frac{1}{3}$
Equation: $y=-\frac{1}{3} x+b$.
To find $b$, substitute $\left(-\frac{3}{2}, \frac{7}{2}\right)$ in the above equation.

$$
\begin{aligned}
& \frac{7}{2}=-\frac{1}{3} \cdot-\frac{3}{2}+b=\frac{1}{2}+b \\
& b=\frac{7}{2}-\frac{1}{2}=3
\end{aligned}
$$

The equation of the perpendicular
bisector of $\overline{A B}$ is $y=-\frac{1}{3} x+3$

Find the intersection point by substitution:
$y=8 x-\frac{19}{2}=-\frac{1}{3} x+3$
$8 x+\frac{1}{3} x=3+\frac{19}{2}$
$\frac{25}{3} x=\frac{25}{2} ; x=\frac{3}{2}$.
Substitute $\frac{3}{2}$ for $x$ in $y=-\frac{1}{3} x+3 . y=-\frac{1}{3} \cdot \frac{3}{2}+3=-\frac{1}{2}+3=\frac{5}{2}$.
$x=\frac{3}{2}, y=\frac{5}{2}$.
Check in the other equation: $y=8 x-\frac{19}{2} .8 \cdot \frac{3}{2}-\frac{19}{2}=\frac{24}{2}-\frac{19}{2}=\frac{5}{2}$
The circumcenter is $\left(\frac{3}{2}, \frac{5}{2}\right)$.

## Centroid:

Median from $A$ to $\overline{B C}$ :
Midpoint of $\overline{B C}:\left(1,-\frac{3}{2}\right)$
A: $(0,8)$, which is also the y-intercept.
Slope from $(0,8)$ to $\left(1,-\frac{3}{2}\right)$ :

Median from $C$ to $\overline{A B}$ :
Midpoint of $\overline{A B}:\left(-\frac{3}{2}, \frac{7}{2}\right)$
$C:(5,-2)$
Slope from $\left(-\frac{3}{2}, \frac{7}{2}\right)$ to $(5,-2)$ :

$$
\frac{8-\left(-\frac{3}{2}\right)}{0-1}=\frac{\frac{19}{2}}{-1}=-\frac{19}{2}
$$

The equation of the median from $A$ to $\overline{B C}$ is $y=-\frac{19}{2} x+8$

$$
\begin{aligned}
& \frac{-2-\frac{7}{2}}{5-\left(-\frac{3}{2}\right)}=\frac{\frac{-11}{2}}{\frac{13}{2}}=-\frac{11}{13} \\
& y=-\frac{11}{13} x+b .
\end{aligned}
$$

Substitute (5, -2) to solve for $b$.
$-2=-\frac{11}{13} \cdot 5+b$
Simplifying: $b=\frac{29}{13}$.
The equation of the median from $C$ to $\overline{A B}$ is $y=-\frac{11}{13} x+\frac{29}{13}$.

Find the intersection point by substitution:
$-\frac{19}{2} x+8=-\frac{11}{13} x+\frac{29}{13}$.
$-\frac{19}{2} x+\frac{11}{13} x=\frac{29}{13}-8$.
$-\frac{225}{26} x=-\frac{150}{26}$.
$x=\frac{2}{3}$. Substitute $x=\frac{2}{3}$ in $y=-\frac{19}{2} x+8$.
$y=-\frac{19}{2} \cdot \frac{2}{3}+8=\frac{5}{3}$.
Check in the other equation: $y=-\frac{11}{13} x+\frac{29}{13}=-\frac{11}{13} \cdot \frac{2}{3}+\frac{29}{13}=-\frac{22}{39}+\frac{87}{39}=\frac{65}{39}=\frac{5}{3}$

The centroid is $\left(\frac{2}{3}, \frac{5}{3}\right)$.
Orthocenter:
Slope of $\overline{B C}: \frac{-1-(-2)}{-3-5}=\frac{1}{-8}$.

Perpendicular slope $=8$.
Slope of $\overline{A B}: \frac{8-(-1)}{0-(-3)}=\frac{9}{3}=3$.
Perpendicular slope $=-\frac{1}{3}$
$A(0,8)$ is also the $y$-intercept.

The equation of the altitude from $A$ to $\overline{B C}$ is $y=8 x+8$.

Use $C(5,-2)$ and slope $=-\frac{1}{3}$ to find the
equation $y=-\frac{1}{3} x+b$.
Substitute C (5, -2).
$-2=-\frac{1}{3} \cdot 5+b$.
$b=-2+\frac{5}{3}=-\frac{1}{3}$.
The equation of the altitude from $C$ to $\overline{A B}$ is $y=-\frac{1}{3} x-\frac{1}{3}$.
Substituting: $8 x+8=-\frac{1}{3} x-\frac{1}{3}$
$8 x+\frac{1}{3} x=-8-\frac{1}{3}$.
$\frac{25}{3} x=-\frac{25}{3}$.
$x=-1$.
Substitute $x=-1$ in $y=8 x+8=-8+8=0$.
Check $y=-\frac{1}{3} x-\frac{1}{3}=0$.
The orthocenter is $(-1,0)$.
Euler Line:
Use the orthocenter $(-1,0)$ and the circumcenter $\left(\frac{3}{2}, \frac{5}{2}\right)$.
Slope: $\frac{\frac{5}{2}-0}{\frac{3}{2}-(-1)}=\frac{\frac{5}{2}}{\frac{5}{2}}=1$.
$U \operatorname{sing}(-1,0), y=0=1(-1)+b=-1+b$.
$b=1$.
Euler Line: $y=x+1$.
Ask participants to add the new terms centroid, circumcenter, Euler line, incenter, and orthocenter to their glossaries.

To close the unit, discuss the van Hiele levels represented in the activity. In $1-3$ participants are performing at the Descriptive Level since participants use properties relating to reflectional symmetry. In the remainder of the activity, $4-12$, participants apply properties at the Descriptive Level, including some Relational Level elements, such as justifications for congruence of segments relating to the circumcenter and incenter. In finding the equation of the Euler Line, participants perform at the Relational Level with respect to linear functions, since various properties of linear properties are interrelated.

## The Meeting Place

## Part 1: Constructions with Patty Paper

The following constructions will be used in Part 2.

1. Perpendicular Bisector of a Segment

Draw a segment on a sheet of patty paper. Label the endpoints $A$ and $B$. By folding, construct the perpendicular bisector of the segment. Mark a point anywhere on the perpendicular bisector. Label the point $C$. Draw $\overline{A C}$ and $\overline{B C}$. Measure $A C$ and $B C$. Write a conjecture relating $C$ to $\overline{A B}$.
2. Perpendicular to a Segment Through a Point Not on the Segment

Draw a segment on patty paper. Mark a point, $E$, not on the segment. By folding, construct a perpendicular line to the segment, passing through $E$.
3. Angle Bisector

Draw an angle, labeled $\angle F G H$ on patty paper. By folding, construct the angle bisector. Mark a point, $I$, on the angle bisector. Using another sheet of patty paper for a right angle tool, or by folding, construct the perpendicular lines from $I$ to the sides of the angle, $\overrightarrow{G F}$ and $\overrightarrow{G H}$. Write a conjecture relating the distance from a point on the angle bisector to the sides of the angle.

## Part 2: Points of Concurrency

4. On a new sheet of patty paper, draw a scalene triangle covering at least half of the sheet. Trace the triangle onto four more sheets, so that you have five congruent triangles. Label each $\triangle X Y Z$.
5. On the first triangle, construct the perpendicular bisectors of the three sides by folding. Label the point of concurrency $C$. Measure and compare the distances from $C$ to the vertices of the triangle. Explain why these distances must be equal. Draw the circumscribed circle, centered at $C$, through the vertices of the triangle. Point $C$ is called the circumcenter. Label the sheet "Circumcenter - Perpendicular Bisectors."
6. On the second triangle, construct the three angle bisectors by folding. Label the point of concurrency $I$. Measure and compare the perpendicular distances from $I$ to the sides of the triangle. Mark the points where the
perpendiculars from $I$ intersect the sides of the triangle. Explain why these distances must be equal. Draw the inscribed circle, centered at $I$, through the marked points on the sides of the triangle. Point $I$ is called the incenter. Label the sheet "Incenter - Angle Bisectors."
7. On the third triangle, by folding, pinch the midpoints of each of the sides of the triangle. With a ruler, draw the three medians, the segments connecting each vertex to the midpoint of the opposite sides. Label the point of concurrency $M$. Measure and determine the ratio into which $M$ divides each median. Label the sheet "Centroid - Medians."
8. On the fourth triangle, by folding, construct the three altitudes (the perpendicular segments from each vertex to the opposite side). If you have an obtuse triangle you will need to extend two of the sides outside the triangle. Label the point of concurrency $O$. Label the sheet "Orthocenter - Altitudes."
9. Place the fifth triangle directly on top of the first triangle, so that the vertices coincide exactly. Mark and label point $C$. Repeat with the other three points of concurrency, $I, M$, and $O$. Which three of the four points are collinear (lie on the same line)? Draw in the line connecting these three points. Label the sheet "Euler Line." The Euler line is named after Leonhard Euler, a Swiss mathematician (1708-1783), who proved that three points of concurrency are collinear.

## Part 3: Algebraic Connections

The points of concurrency can be found algebraically using knowledge of slope, $y$-intercept, perpendicular lines, the writing of linear equations, and the solving of systems of linear equations.

In the following problems, algebraically find the circumcenter, centroid, orthocenter, and the equation of the Euler line.
10. Triangle $A B C$ has vertices $A(0,8), B(-3,-1)$, and $C(5,-2)$.
11. Triangle $D E F$ has vertices $D(-2,0), E(-6,-2)$, and $F(3,-5)$.
12. Triangle $G H J$ has vertices $G(3,7), H(-1,-1)$, and $J(5,-4)$.

## References and Additional Resources

Geddes, D. (1992). Geometry in the middle grades, Addenda Series, Grades 5-8 (F. R. Curcio, Ed.). Reston, VA: National Council of Teachers of Mathematics.

Kinsey, L. C., \& Moore, T. E. (2002). Symmetry, shape, and space: An introduction to mathematics through geometry. Emeryville, CA: Key College Publishing.

National Council of Teachers of Mathematics (2000). Principles and standards for school mathematics. Reston, VA: National Council of Teachers of Mathematics.

Serra, M. (2003). Discovering geometry: An investigative approach ( $3^{\text {rd }}$ ed.). Emeryville, CA: Key Curriculum Press.

## Unit 3 - Quadrilaterals

## Isosceles Right Triangle Reflections

Overview: $\quad$ Participants develop the properties of squares through reflections of isosceles right triangles.

## Objective: TExES Mathematics Competencies

III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.013.A. The beginning teacher analyzes the properties of polygons and their components.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
III.014.B. The beginning teacher uses the properties of transformations and their compositions to solve problems.
III.014.D. The beginning teacher applies transformations in the coordinate plane.
V.018.B. The beginning teacher applies correct mathematical reasoning to derive valid conclusions from a set of premises.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.
V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

## Geometry TEKS

b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.
e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.
f.1. The student uses similarity properties and transformations to explore and justify conjectures about geometric figures.

Background: Participants need to have a knowledge of transformations.
Materials: colored pencils, easel paper, colored markers, centimeter ruler, transparency

New Terms: rotational symmetry

## Procedures:

In the Quadrilaterals unit, the properties of triangles and transformations are used to develop the properties of quadrilaterals.

Participants work the activity individually but verify solutions informally with group members. Facilitate by providing minimal prompts to help participants clarify their thinking. Try not to answer questions directly.

Bertrand Russell said, "...what matters in mathematics...is not the intrinsic nature of our terms but the logical nature of their interrelations." The quadrilateral unit explores the interrelations of triangles and quadrilaterals, using transformations as a tool to construct quadrilaterals.

Mark isosceles right $\triangle A B C$ with the known properties in terms of angles and sides, using tick marks and colors.

Verify the properties of angles and sides with your group members.
The isosceles right triangle has a right angle, $\angle B$, two congruent sides, $\overline{A B}$ and $\overline{B C}$, $\overline{A B} \perp \overline{B C}$, and two congruent angles $\angle A$ and $\angle C$, each $45^{\circ}$.

Reflect $\triangle A B C$ across the line containing $\overline{B C}$. Label image vertices and properties appropriately.

1. What type of triangle is $\triangle A C A^{\prime}$ ? Justify your answer.
$\triangle A C A^{\prime}$ is an isosceles right triangle.
$A C=C A^{\prime}$
$m \angle A C A^{\prime}=m \angle A C B+m \angle A^{\prime} C B=90^{\circ}$.

2. Reflect $\triangle A C A^{\prime}$ and its component parts across the line containing $\overline{A A^{\prime}}$. All of the properties of $\triangle A C A^{\prime}$ apply to the reflected triangle. Label the properties of quadrilateral $A C A^{\prime} C^{\prime}$.

3. Classify the quadrilateral $A C A^{\prime} C^{\prime}$, formed from the composite reflections of an isosceles right triangle.
$A C A^{\prime} C^{\prime}$ is a square. The figure has four right angles and four congruent sides.
4. List the properties of quadrilateral $A C A^{\prime} C^{\prime}$ in terms of the sides, angles, diagonals and symmetry in the table below.

Possible properties follow.
Sides:

- All four sides are congruent.
- Opposite sides are parallel because alternate interior angles are congruent.

Using symbols, $\overline{A C} \| \overline{A^{\prime} C^{\prime}}$ and $\overline{C A^{\prime}} \| \overline{A C^{\prime}}$.

- Consecutive sides are perpendicular, because vertex angles of the figure are all right angles, for example, $\overline{A C^{\prime}} \perp \overline{A^{\prime} C^{\prime}}$.

Vertex angles:

- All four angles are congruent right angles.
- The vertex angles are bisected by the diagonals.
- Opposite angles are congruent and supplementary.
- Consecutive angles are congruent and supplementary.


## Diagonals:

- There are two diagonals.
- Diagonals are congruent to each other.
- Diagonals bisect each other at right angles.
- Diagonals bisect the vertex angles of the square.
- Diagonals lie on two of the lines of symmetry for the figure.


## Symmetry:

- There are four lines of symmetry.
- The diagonals lie on two of the lines of symmetry.
- The other two lines of symmetry pass through the midpoints of the sides of the square.
- $90^{\circ}$ (4-fold) rotational symmetry exists. The figure can be rotated $90^{\circ}$ so that the resulting image coincides with the original image. When the figure is rotated four times through $90^{\circ}$, the original vertices coincide.

In general, a figure has rotational symmetry if there is a rotation that results in the image superimposing on the pre-image. Remind participants to add the new term rotational symmetry to their glossaries.

To complete the activity, each group draws the figure and lists its properties on a sheet of easel paper. The sheet of easel paper is placed on the wall for a gallery walk. Pairs of groups view each other's work. Allow groups about 5 minutes to meet and discuss any differences or errors on the posters.

Bring participants together for a whole class discussion. Summarize the properties of squares. Leave the posters on the wall. At the conclusion of the quadrilateral unit, participants can compare the properties of the square, rhombus, kite, rectangle, parallelogram and trapezoid.

Participants are performing at the van Hiele Descriptive Level because they develop properties of squares.

## Isosceles Right Triangle Reflections

Bertrand Russell said, "...what matters in mathematics...is not the intrinsic nature of our terms but the logical nature of their interrelations." The quadrilateral unit explores the interrelations of triangles and quadrilaterals, using transformations as a tool to construct quadrilaterals.

Mark isosceles right $\triangle A B C$ with the known properties in terms of angles and sides, using tick marks and colors.

|  |  |  |  |  |  | $C$ |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  |  |  |  |  |  |  |  |  |  |  |  |

Verify the properties of angles and sides with your group members. Reflect $\triangle A B C$ across the line containing $\overline{B C}$, and label image vertices and properties appropriately.

1. What type of triangle is $\triangle A C A^{\prime}$ ? Justify your answer.
2. Reflect $\triangle A C A^{\prime}$ and its component parts across the line containing $\overline{A A^{\prime}}$. All of the properties of $\triangle A C A^{\prime}$ apply to the reflected triangle. Label the properties of quadrilateral $A C A^{\prime} C^{\prime}$.
3. Classify the quadrilateral $A C A^{\prime} C^{\prime}$, formed from the composite reflections of an isosceles right triangle.
4. List the properties of quadrilateral $A C A^{\prime} C^{\prime}$ in terms of the sides, angles, diagonals and symmetry in the table below.

| Sides | Vertex Angles |
| :---: | :---: |
|  |  |
| Diagonals |  |
|  |  |

## Scalene Right Triangle Reflections

Overview: Participants develop the properties of rhombi through reflections of scalene right triangles.

## Objective: TExES Mathematics Competencies

III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.013.A. The beginning teacher analyzes the properties of polygons and their components.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
III.014.B. The beginning teacher uses the properties of transformations and their compositions to solve problems.
III.014.D. The beginning teacher applies transformations in the coordinate plane.
V.018.B. The beginning teacher applies correct mathematical reasoning to derive valid conclusions from a set of premises.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.
V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

## Geometry TEKS

b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.D. The student uses inductive reasoning to formulate a conjecture. c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties of parallel and perpendicular lines.
e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.
f.1. The student uses similarity properties and transformations to explore and justify conjectures about geometric figures.

Background: Participants need to have a knowledge of transformations.
Materials: colored pencils, easel paper, graph paper, colored markers, centimeter ruler

## New Terms:

## Procedures:

Distribute the activity sheet. Ask participants to complete the activity individually but informally verify with group members. While participants work, walk around the room listening and facilitating. Try not to directly answer questions, but rather provide minimal prompts to help participants clarify their thinking.

When the group members complete the activity, each group draws the figure and lists its properties on a sheet of easel paper. The sheet of easel paper is placed on the wall for a gallery walk. Allow pairs of groups about 5 minutes to meet and discuss any differences or errors on the posters.

In this activity, a scalene right triangle is reflected over the line containing one of the legs of the triangle, then the composite figure is reflected over the line containing the other leg. Predict which quadrilateral will be created.

In the middle of a sheet of graph paper, draw a scalene right triangle, $\triangle A B C$, with the right angle at $B$. Make sure that the legs of the triangle lie along grid lines, and that the vertices are located on grid line intersections. The legs of the triangle should be 2 to 3 inches long. Mark the sides and angles with known properties of scalene right triangles. Use different tick marks to indicate non-congruency.

Reflect $\triangle A B C$ across the line containing $\overline{B C}$. Label image vertices with prime marks.

1. What type of triangle is $\triangle A C A^{\prime}$ ? Why? Discuss with your group to make sure there is agreement.
$\triangle A C A^{\prime}$ is an isosceles acute triangle or an isosceles obtuse triangle. The congruent legs and congruent base angles are formed as a result of the reflection. The base angles are not $45^{\circ}$ angles, and the vertex angle at $C$ is not a right angle.

2. Reflect $\triangle A C A^{\prime}$ and its component parts across the line containing $\overline{A A^{\prime}}$. Label the properties of quadrilateral $A C A^{\prime} C^{\prime}$. Discuss with your group to make sure there is agreement.
3. Classify the quadrilateral $A C A^{\prime} C^{\prime}$. Justify.

Quadrilateral $A C A^{\prime} C^{\prime}$ is a rhombus because it has four congruent sides.

4. List the properties of quadrilateral $A C A^{\prime} C^{\prime}$ in terms of the sides, angles, diagonals and symmetry in the table below.

## Sides:

- All four sides are congruent.
- Opposite sides are parallel because alternate interior angles are congruent.

Vertex Angles:

- Opposite angles are congruent.
- Consecutive angles are supplementary because the pre-image acute angles are complementary. The consecutive angles, which are composed of two sets of the acute complementary angles, must be supplementary.

Diagonals:

- There are two diagonals.
- Diagonals are not congruent to each other.
- Diagonals bisect each other.
- Diagonals intersect at $90^{\circ}$ angles.
- Diagonals bisect the vertex angles of the rhombus, because the vertex angles were formed by reflection.
- Diagonals are lines of symmetry of the figure, because they lie on the original reflection lines.

Symmetry:

- There are two lines of symmetry.
- The diagonals lie on the two lines of symmetry, passing through opposite vertices of the rhombus.

Bring participants together for a whole class discussion. Compare the properties of the rhombus with the properties of the square. Possible comparisons follow:

- Both figures have four congruent sides.
- In both quadrilaterals opposite vertex angles are congruent and consecutive vertex angles are supplementary.
- In both quadrilaterals the diagonals bisect each other at right angles, and bisect the vertex angles.
- In both quadrilaterals the diagonals lie on lines of symmetry.

As the properties of each quadrilateral are listed, the posters remain on the wall so that the properties of squares, kites, rectangles, parallelograms and trapezoids can be compared and contrasted.

Participants are performing at the van Hiele Descriptive Level because properties of a rhombus are being developed. In the discussion comparing the properties of the square and rhombus participants approach the Relational Level.

## Scalene Right Triangle Reflections

In this activity, a scalene right triangle is reflected over the line containing one of the legs of the triangle, then the composite figure is reflected over the line containing the other leg. Predict which quadrilateral will be created.

In the middle of a sheet of graph paper, draw a scalene right triangle, $\triangle A B C$, with the right angle at $B$. Make sure that the legs of the triangle lie along grid lines, and that the vertices are located on grid line intersections. The legs of the triangle should be 2 to 3 inches long. Mark the sides and angles with known properties of scalene right triangles. Use different tick marks to indicate non-congruency.

Reflect $\triangle A B C$ across the line containing $\overline{B C}$. Label image vertices with prime marks.

1. What type of triangle is $\triangle A C A^{\prime}$ ? Why? Discuss with your group to make sure there is agreement.
2. Reflect $\triangle A C A^{\prime}$ and its component parts across the line containing $\overline{A A^{\prime}}$. Label the properties of quadrilateral $A C A^{\prime} C^{\prime}$. Discuss with your group to make sure there is agreement.
3. Classify the quadrilateral $A C A^{\prime} C^{\prime}$. Justify.
4. List the properties of quadrilateral $A C A^{\prime} C^{\prime}$ in terms of the sides, angles, diagonals and symmetry in the table below.

| Sides | Vertex Angles |
| :---: | :---: |
|  |  |
| Diagonals |  |
|  |  |

## Scalene Acute/Obtuse Triangle Reflections

Overview: Participants develop the properties of kites through reflections of scalene acute or scalene obtuse triangles.

## Objective: TExES Mathematics Competencies

III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.013.A. The beginning teacher analyzes the properties of polygons and their components.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
III.014.D. The beginning teacher applies transformations in the coordinate plane.
V.018.B. The beginning teacher applies correct mathematical reasoning to derive valid conclusions from a set of premises. V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.
V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

## Geometry TEKS

b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.
e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.
f.1. The student uses similarity properties and transformations to explore and justify conjectures about geometric figures.

Background: Participants need to have a knowledge of transformations for this activity.

Materials: colored pencils, easel paper, colored markers, centimeter ruler

## New Terms:

## Procedures:

Distribute the activity sheet. Participants work the activity individually, but informally verify with group members. Facilitate by providing minimal prompts to help participants clarify their thinking. Try not to answer questions directly.

Each group draws the figure and lists its properties on a sheet of easel paper. When completed the sheet of easel paper is posted on the wall for a gallery walk. Give pairs of groups about 5 minutes to meet and discuss any differences or errors on the posters.

In this activity, a scalene acute triangle or a scalene obtuse triangle is reflected across the line containing one of its sides. Predict what quadrilateral will be created.

In your group decide who will draw a scalene acute triangle and who will draw a scalene obtuse triangle. In the middle of a sheet of graph paper, draw $\triangle A B C$ so that $\overline{B C}$ coincides with a grid line and all three vertices lie at grid line intersections. The sides of the triangle should be 2 to 3 inches long. Mark the sides and angles using different tick marks to indicate non-congruency.

Possible examples are shown.


Reflect $\triangle A B C$ across the line containing $\overline{B C}$, and label the image appropriately.

1. Classify quadrilateral $A C A^{\prime} B$. Justify. $A C A$ 'B is a kite. A kite is a quadrilateral with exactly two distinct pairs of congruent consecutive sides. The term diamond is sometimes used at the Visual Level.

2. List the properties of quadrilateral $A C A^{\prime} B$ in terms of the sides, angles, diagonals and symmetry in the table below.

Sides:

- Two pairs of sides are congruent.
- Two sets of consecutive sides are congruent.
- Opposite sides are not congruent.

Note: Opposite sides are not parallel because alternate interior angles are not congruent.

Vertex angles:

- Only one pair of opposite angles is congruent. One pair of opposite angles is not congruent.
- Consecutive angles are not congruent.
- The non-congruent vertex angles are bisected by one of the diagonals. The congruent vertex angles are not bisected by a diagonal.


## Is it possible for a kite to have a pair of right angles?

 The figures for this activity were constructed from acute or obtuse triangles, but if a scalene right triangle is reflected across the line containing its hypotenuse, then the congruent pair of vertex angles are right angles.

Diagonals:

- There are two diagonals.
- One diagonal lies on the line of symmetry.
- The diagonal lying on the line of symmetry bisects the other diagonal at right angles.
- Diagonals may not be congruent to each other.

Is it possible for the kite to have congruent diagonals? Yes. The figure to the right is an example of a kite with congruent diagonals.


Note: Ask participants to look at the triangles formed on either side of $\overline{A A^{\prime}}$, the diagonal which does not lie on the line of symmetry. These triangles, $\triangle \mathrm{ABA}^{\prime}$ and $\Delta \mathrm{ACA}^{\prime}$, are both isosceles triangles.

The kites below are both convex.


The kite below is concave.


## Symmetry:

- There is one line of symmetry.
- The line of symmetry contains one of the diagonals.

Bring participants together for a whole class discussion. Summarize the properties of kites.

Participants are performing at the van Hiele Descriptive Level because they develop properties of a kite.

## Scalene Acute/Obtuse Triangle Reflections

In this activity a scalene acute triangle or a scalene obtuse triangle is reflected across the line containing one of its sides. Predict what quadrilateral will be created.

In your group decide who will draw a scalene acute triangle and who will draw a scalene obtuse triangle. In the middle of a sheet of graph paper, draw $\triangle A B C$ so that $\overline{B C}$ coincides with a grid line and all three vertices lie at grid line intersections. The sides of the triangle should be 2 to 3 inches long. Mark the sides and angles using different tick marks to indicate noncongruency.

Reflect $\triangle A B C$ across the line containing $\overline{B C}$, and label the image appropriately.

1. Classify quadrilateral $A C A^{\prime} B$. Justify.
2. List the properties of quadrilateral $A C A^{\prime} B$ in terms of the sides, angles, diagonals and symmetry in the table below.

| Sides | Vertex Angles |
| :---: | :---: |
|  |  |
| Diagonals |  |
|  |  |
|  |  |

## Rotate a Triangle

Overview: Participants discover properties of rectangles by rotating a right triangle around the midpoint of its hypotenuse, and discover the properties of parallelograms by rotating a non-right triangle around the midpoint of one of its sides.

## Objective: TExES Mathematics Competencies

III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.013.A. The beginning teacher analyzes the properties of polygons and their components.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
III.014.D. The beginning teacher applies transformations in the coordinate plane.
V.018.B. The beginning teacher applies correct mathematical reasoning to derive valid conclusions from a set of premises. V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.
V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

## Geometry TEKS

b.2.B. Makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of circles and the lines that intersect them.
e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.
f.1. The student uses similarity properties and transformations to explore and justify conjectures about geometric figures.

Background: Participants need a knowledge of the properties of rotation and parallel lines.

Materials: easel paper, graph paper, colored markers, patty paper, centimeter ruler

## New Terms:

## Procedures:

Distribute the activity sheets. Participants work on all seven items in groups. During the whole class discussion, ask two volunteers to record properties of rectangles and parallelograms on separate sheets of easel paper.

1. In the middle of a sheet of graph paper draw scalene right $\triangle A B C$, with the right angle at vertex $C$. Draw the legs along grid lines and locate the vertices at grid line intersections. The lengths of the legs of the triangles should be 2 to 3 inches long. Locate the midpoint, $M$, of the hypotenuse. Draw the median to the hypotenuse. Label the figure appropriately to indicate congruence or non-congruence.
2. Rotate $\triangle A B C 180^{\circ}$ around $M$. Label the image appropriately.

The figures below represent the process.

3. In your group discuss and list the properties of rectangle $A C B C^{\prime}$.

The following is a list of possible properties for the rectangle:
Sides:

- Opposite sides are congruent. (Rotation preserves congruence.)
- Opposite sides are parallel. (The consecutive angles are supplementary. If the interior angles on the same side of a transversal are congruent, then the lines intersected by the transversal are parallel.)
- Consecutive sides are perpendicular to each other.

Vertex angles:

- All vertex angles are congruent right angles.
- Opposite angles are congruent and supplementary.
- Consecutive angles are congruent and supplementary.

Diagonals:

- Diagonals bisect each other.
- Diagonals are congruent.
- The point of intersection of the diagonals is also the center of the circumscribed circle.


Symmetry:

- The rectangle has two symmetry lines.

- The rectangle has $180^{\circ}$ (or 2-fold) rotational symmetry.

Participants may argue that the diagonals are symmetry lines. To clear up this misconception, trace the figure on patty paper and fold along the diagonals.
4. On a clean sheet of graph paper draw obtuse or acute scalene $\triangle A B C$, with one of its sides along one of the grid lines. Locate the vertices at grid line intersections. The sides of the triangle should be 1.5 to 3 inches long. Locate the midpoint, $M$, of $\overline{A B}$. Draw the median to side $\overline{A B}$. Label the figure appropriately indicating congruence or non-congruence.
5. Rotate $\triangle A B C 180^{\circ}$ around $M$. Label the image appropriately.

The figures below represent examples.

6. In your group discuss and list the properties of parallelogram $A C B C^{\prime}$ under the given headings.

Sides:

- Opposite sides are congruent. (Rotation preserves congruence.)
- Opposite sides are parallel. (Alternate interior angles are congruent, because rotation preserves congruence.)

Vertex angles:

- Opposite vertex angles are congruent. (Rotation preserves congruence.)
- Consecutive angles are supplementary. (Interior angles on the same side of a transversal that intersects parallel lines are supplementary.)


## Diagonals:

- Diagonals bisect each other. ( $M$ is the midpoint of $\overline{A B}$ and $\overline{M C}$ is mapped to $M C^{\prime}$, so that $M$ the midpoint of $C C^{\prime}$.)

Symmetry:

- The parallelogram has $180^{\circ}$ (or 2-fold) rotation. (The figure was produced using $180^{\circ}$ rotation.)

Participants may argue that the diagonals are symmetry lines. To clear up this misconception, trace the figure on patty paper and fold along the diagonals.
7. Compare the properties of the parallelogram with the properties of the rectangle.

In both quadrilaterals opposite sides are parallel and congruent; opposite angles are congruent; diagonals bisect each other.

Application problems:
8. Calculate the measure of each lettered angle.

9. $\overline{R C}$ is a diagonal of rectangle $R E C T$. Where can the other two vertices, $E$ and $T$, be located?
The diagonals are congruent and intersect each other at their respective midpoints. The other diagonal can be any congruent line segment, whose midpoint is also the midpoint of the given segment.
Find the midpoint of RC. Draw a segment from the midpoint to a point not on $\overline{R C}$, congruent to one half of $\overline{R C}$. Extend the segment an equal distance on the opposite side of the midpoint.


Alternatively: Draw a circle using $\overline{R C}$ as the diameter. $\overline{E T}$ is also a diameter.


Participants are performing at the van Hiele Descriptive Level because properties of rectangles and parallelograms are being developed. The comparison of the properties of the rectangle and parallelogram approaches the Relational Level.

In 7, participants apply the properties of parallelograms and rectangles at the Descriptive Level.

In 8, the first solution requires the Descriptive Level. In the second solution, participants combine two figures with related properties, thus approaching the Relational Level.

## Rotate a Triangle

1. In the middle of a sheet of graph paper draw scalene right $\triangle A B C$, with the right angle at vertex $C$. Draw the legs along grid lines and locate the vertices at grid line intersections. The lengths of the legs of the triangle should be 2 to 3 inches long. Locate the midpoint, $M$, of the hypotenuse. Draw the median to the hypotenuse. Label the figure appropriately to indicate congruence or non-congruence.
2. Rotate $\triangle A B C 180^{\circ}$ around $M$. Label the image appropriately.
3. In your group discuss and list the properties of rectangle $A C B C^{\prime}$.

| Sides | Vertex Angles |
| :---: | :---: |
|  |  |
| Diagonals |  |
|  | Symmetry |
|  |  |
|  |  |
|  |  |

4. On a clean sheet of graph paper draw obtuse or acute scalene $\triangle A B C$, with one of its sides along one of the grid lines. Locate the vertices at grid line intersections. The sides of the triangle should be 1.5 to 3 inches long. Locate the midpoint, $M$, of $\overline{A B}$. Draw the median to side $\overline{A B}$. Label the figure appropriately indicating congruence or non-congruence.
5. Rotate $\triangle A B C 180^{\circ}$ around $M$. Label the image appropriately.
6. In your group discuss and list the properties of parallelogram $A C B C^{\prime}$ under the given headings.

| Sides | Vertex Angles |
| :--- | :---: |
|  |  |
| Diagonals |  |
|  | Symmetry |
|  |  |

7. Compare the properties of the parallelogram with the properties of the rectangle.

Application problems:
8. Calculate the measure of each lettered angle.


$$
a=\ldots \quad c=\ldots \quad d=\ldots \quad e=
$$

$$
f=\quad g=\quad \quad h=\quad \quad j=\ldots \quad k=
$$

9. $\overline{R C}$ is a diagonal of rectangle $R E C T$. Where can the other two vertices, $E$ and $T$, be located?


## Truncate a Triangle's Vertex

Overview: Participants discover the properties of trapezoids.

## Objective: TExES Mathematics Competencies

II.006.B. The beginning teacher writes equations of lines given various characteristics (e.g., two points, a point and slope, slope and $y$ intercept).
II.006.G. The beginning teacher models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology.
III.011.A. The beginning teacher applies dimensional analysis to derive units and formulas in a variety of situations (e.g., rates of change of one variable with respect to another) and to find and evaluate solutions to problems.
III.011.B. The beginning teacher applies formulas for perimeter, area, surface area, and volume of geometric figures and shapes (e.g., polygons, pyramids, prisms, cylinders, cones, spheres) to solve problems.
III.012.B. The beginning teacher uses properties of points, lines, planes, angles, lengths, and distances to solve problems.
III.012.C. The beginning teacher applies the properties of parallel and perpendicular lines to solve problems.
III.012.E. The beginning teacher describes and justifies geometric constructions made using compass and straightedge, reflection devices, and other appropriate technologies.
III.013.A. The beginning teacher analyzes the properties of polygons and their components.
III.014.E. The beginning teacher applies concepts and properties of slope, midpoint, parallelism, perpendicularity, and distance to explore properties of geometric figures and solve problems in the coordinate plane.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.
V.018.F. The beginning teacher evaluates how well a mathematical model represents a real-world situation.
V.019.B. The beginning teacher understands how mathematics is used to model and solve problems in other disciplines (e.g., art, music, science, social science, business).
V.019.D. The beginning teacher communicates mathematical ideas using a variety of representations (e.g., numeric, verbal, graphical, pictorial, symbolic, concrete).
V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

## Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.D. The student uses inductive reasoning to formulate a conjecture.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.
d.2.B. The student uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons.
d.2.C. The student develops and uses formulas including distance and midpoint.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.

Background: Participants need a knowledge of properties of isosceles triangles, parallel lines, and their associated angles.

Materials: easel paper, graph paper, colored markers, centimeter ruler
New Terms: isosceles trapezoid, midsegment

## Procedures:

Distribute the activity sheet. Participants complete $1-8$. They may work independently or in groups. 1 and 2 provide information and vocabulary needed to define a trapezoid and an isosceles trapezoid.

1. Draw a large scalene triangle. Label it $\triangle M R A$. Locate a point $T$ on $\overline{M R}$. Construct a line parallel to $\overline{R A}$, through point $T$, which intersects $\overline{M A}$ at point $P . T R A P$ is a trapezoid. The parallel sides $R A$ and $T P$ are called the bases of the trapezoid. In your figure clearly identify the angle relations within the trapezoid.

2. Draw a large isosceles triangle. Label it $\triangle Q S O$ with vertex angle $\angle Q$. Locate a point $I$ on $\overline{S Q}$. Construct a line parallel to side $\overline{S O}$ through point $I$, which intersects $\overline{Q O}$ at point C. ISOC is an isosceles trapezoid. The congruent pairs of angles in an isosceles trapezoid are called base angles. In your figure, clearly identify the angle relations within the trapezoid.


An isosceles trapezoid is a trapezoid with congruent legs.
3. Work with your group to find the angle and side properties of trapezoids and isosceles trapezoids, using your knowledge of parallel lines, isosceles and scalene triangles. Be prepared to share and justify the properties during whole class discussion. Note that some books define a trapezoid as a quadrilateral with "at least one pair of parallel sides." Others define a trapezoid as a quadrilateral with "only one pair of parallel sides". This module will use the latter definition.

Possible properties:

- A trapezoid has one pair of parallel sides.
- The sum of the angles of a trapezoid is $360^{\circ}$.
- The pairs of consecutive angles at opposite bases are supplementary. (Interior angles on the same side of a transversal that intersects parallel lines are supplementary.)
- In an isosceles trapezoid the base angles (two pairs) are congruent.
- In an isosceles trapezoid the non-parallel sides are congruent.

4. Locate the vertices of quadrilateral $A B C D$ on the coordinate plane at $A(-7,-3)$, $B(-3,3), C(1,5)$, and $D(5,3)$. Explain why $A B C D$ is a trapezoid.


Sides $\overline{B C}$ and $\overline{A D}$ are parallel because their slopes are both $\frac{1}{2}$. The other two sides have unequal slopes.
5. Find the midpoints, $M$ and $L$, of $\overline{A B}$ and $\overline{C D}$ respectively. $\overline{M L}$ is called a midsegment. $M(-5,0), L(3,4)$.

A midsegment is a line segment with endpoints that are the midpoints of the legs of the trapezoid.
6. How do the coordinates of a midpoint of a segment relate to the coordinates of its endpoints?
The coordinates of the midpoint are the averages of the coordinates of the endpoints:
$M(-5,0)=\left(\frac{-7-3}{2}, \frac{-3+3}{2}\right) ; L(3,4)=\left(\frac{1+5}{2}, \frac{5+3}{2}\right)$.
7. How does the slope of the midsegment relate to the slopes of the parallel sides of the trapezoid?
The slope of the midsegment is $\frac{1}{2}$, which is the same as the slopes of the bases.
Therefore the midsegment is parallel to the bases.
8. How does the length of the midsegment relate to the lengths of the parallel sides of the trapezoid?
The length of midsegment $\overline{M L}$ is the average of the lengths of $\overline{B C}$ and $\overline{A D}$.
When most groups have completed $1-8$, conduct a whole class discussion on the properties of trapezoids. Participants justify each property. Possible properties, with justifications in parentheses follow:

- A trapezoid has exactly one pair of parallel sides (shown by the construction).
- The sum of the angles is $360^{\circ}$. (There are two sets of interior supplementary angles between parallel lines.)
- The pairs of consecutive angles at opposite bases are supplementary. (The interior angles on the same side of a transversal that intersects parallel lines are supplementary.)
- In an isosceles trapezoid, the two pairs of base angles are congruent. (The two base angles from the original isosceles triangle are congruent; the other two angles are supplementary to the original two base angles, and must also be congruent to each other.)
- In an isosceles trapezoid, the non-parallel sides are congruent. (Using 2 as an example, the base angles are congruent and congruent to the corresponding base angles of $\triangle Q I C$. Therefore, $\triangle Q I C$ is isosceles and $\overline{Q I} \cong \overline{Q C}$. Since $\overline{Q S} \cong \overline{Q O}$, the congruent sides of $\triangle Q S O$, then by subtraction, $\overline{I S} \cong \overline{C O}$ ).
- The midsegment of a trapezoid is the segment connecting the midpoints of the nonparallel sides.
- The midsegment of a trapezoid is parallel to the bases. Its length is the average of the lengths of the two bases.

Frequently the bases of a trapezoid are designated by the variables $b_{1}$ and $b_{2}$. Write an expression for the length of the midsegment in terms of $b_{1}$ and $b_{2}$.
The length of the midsegment is given by $\frac{b_{1}+b_{2}}{2}$.
9. Draw a triangle on a coordinate grid. Locate the midpoints of two of the sides. Draw the midsegment. Why is the midsegment parallel to the third side?
In the example shown, the slopes of the base and the midsegment are both $\frac{2}{3}$, and thus the base and the midsegment are parallel.


Compare the length of the midsegment to the length of the parallel side of the triangle.

The length of the midsegment is half of the length of the parallel side.

Use the formula for the length of the midsegment of a trapezoid to justify this relationship.

Since $b$ is the length of the parallel side, or base, of the triangle, then in the formula for the length of the midsegment, $\frac{b_{1}+b_{2}}{2}, b_{1}=0, b_{2}=b$. By substitution, the length of the midsegment is $\frac{b}{2}$.

Application problems:
10.

11. The perimeter of isosceles trapezoid $A D E F$ is $218 \mathrm{in} . \overline{B C}$ is the midsegment. Find $A D$.


The non-parallel sides of the trapezoid are congruent. Each non-parallel side measures $2(2 x+1)$. The longer base is 8 in. longer than the shorter base.
Perimeter of $A D E F=2(4 x-1)+8+2[2(2 x+1)]$

$$
=8 x-2
$$

$+8+8 x+4$
$=16 x+10$ $=218 \mathrm{in}$.

13 in.
Therefore, $A D=2(2 x+1)=4 x+2$

$$
\begin{aligned}
& =4(13)+2 \\
& =54 \mathrm{in.}
\end{aligned}
$$

12. In the two-dimensional figure, find the angle measure $x$ and $y$. Explain.


The following problems are taken from Serra, M. (2003). Discovering Geometry: An Investigative Approach ( $3^{\text {rd }}$ ed.). Emeryville, CA: Key Curriculum Press, pp. 271, 283 with permission from Key Curriculum Press.

The Romans used the classical arch design in bridges, aqueducts, and buildings in the early centuries of the Common Era. The classical semicircular arch is really half of a regular polygon built with wedge-shaped blocks whose faces are isosceles trapezoids. Each block supports the blocks surrounding it.
13. The inner edge of the arch in the diagram is half of a regular 18-gon. Calculate the measures of all the angles in the nine isosceles trapezoids making up the arch.


Imagine that the isosceles trapezoids become isosceles triangles by regaining their truncated vertices. There are nine isosceles triangles, whose vertices meet at the center of a semicircle. The nine vertices each contribute $20^{\circ}$ to the $180^{\circ}$ at the center of the span. The sum of the base angles of each isosceles triangle is $180^{\circ}-20^{\circ}=160^{\circ}$. The trapezoid's base angles on the outer edge of the arch each measure $80^{\circ}$. The base angles on the inner edge of the arch are supplementary to the exterior base angle, and measure $100^{\circ}$ each.
14. What is the measure of each angle in the isosceles trapezoid face of a voussoir in a 15 -stone arch?
As in 13, the vertices of the isosceles triangles created by the 15 trapezoids span $180^{\circ}$. Each vertex spans $12^{\circ}$. The sum of the outer base angles in each trapezoid is $180^{\circ}-12^{\circ}=168^{\circ}$. Each outer base angle measures $84^{\circ}$. Each supplementary inner base angle measures $96^{\circ}$.

Remind participants to add the new terms isosceles trapezoid and midsegment to their glossaries.

Participants are performing at the van Hiele Relational Level as they develop properties of trapezoids and isosceles trapezoids using deductive reasoning rather than observation and measurement.

## Truncate a Triangle's Vertex

1. Draw a large scalene triangle. Label it $\triangle M R A$. Locate a point $T$ on $\overline{M R}$. Construct a line parallel to side $\overline{R A}$, through point $T$, which intersects $\overline{M A}$ at $P$. TRAP is a trapezoid. The parallel sides $\overline{R A}$ and $\overline{T P}$ are called the bases of the trapezoid. In your figure clearly identify the angle relations within the trapezoid.
2. Draw a large isosceles triangle. Label it $\triangle Q S O$ with vertex angle $\angle Q$. Locate a point $I$ on $\overline{S Q}$. Construct a line parallel to side $\overline{S O}$, through point $I$, which intersects $\overline{Q O}$ at $C$. ISOC is an isosceles trapezoid. The congruent pairs of angles in an isosceles trapezoid are called base angles. In your figure, clearly identify the angle relations within the trapezoid.
3. Work with your group to find the angle and side properties of trapezoids and isosceles trapezoids, using your knowledge of parallel lines, isosceles and scalene triangles. Be prepared to share and justify your properties during whole class discussion. Note that some books define a trapezoid as a quadrilateral with "at least one pair of parallel sides." Others define a trapezoid as a quadrilateral with "only one pair of parallel sides". This module will use the latter definition.
4. Locate the vertices of quadrilateral $A B C D$ on the coordinate plane at $A(-7,-3), B(-3,3), C(1,5)$, and $D(5,3)$. Explain why $A B C D$ is a trapezoid.

5. Find the midpoints, $M$ and $L$, of $\overline{A B}$ and $\overline{C D}$ respectively. $\overline{M L}$ is called a midsegment.
6. How do the coordinates of a midpoint of a segment relate to the coordinates of its endpoints?
7. How does the slope of the midsegment relate to the slopes of the parallel sides of the trapezoid?
8. How does the length of the midsegment relate to the lengths of the parallel sides of the trapezoid?
9. Draw a triangle on a coordinate grid. Locate the midpoints of two of the sides. Draw the midsegment.
Why is the midsegment parallel to the third side?

Compare the length of the midsegment to the length of the parallel side of the triangle.

Application problems:
10.

11. The perimeter of isosceles trapezoid $A D E F$ is 218 in. $\overline{B C}$ is the midsegment. Find $A D$.

12. In the two-dimensional figure below, find the angle measures $x$ and $y$. Explain.


The following problems are taken from Serra, M. (2003). Discovering Geometry: An Investigative Approach ( $3^{\text {rd }}$ ed.) Emeryville, CA: Key Curriculum Press, pp. 271, 283 with permission from Key Curriculum Press.


The Romans used the classical arch design in bridges, aqueducts, and buildings in the early centuries of the Common Era. The classical semicircular arch is really half of a regular polygon built with wedge-shaped blocks whose faces are isosceles trapezoids. Each block supports the blocks surrounding it.
13. The inner edge of the arch in the diagram is half of a regular 18-gon. Calculate the measures of all the angles in the nine isosceles trapezoids making up the arch.

14. What is the measure of each angle in the isosceles trapezoid face of a voussoir in a 15 -stone arch?

## Vesica Pisces

Overview: Using properties of the different quadrilaterals, participants determine the figures within the vesica pisces.

## Objective: TExES Mathematics Competencies <br> III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems. <br> III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles). <br> V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas. <br> V.018.F. The beginning teacher evaluates how well a mathematical model represents a real-world situation. <br> V.019.B. The beginning teacher understands how mathematics is used to model and solve problems in other disciplines (e.g., art, music, science, social science, business). <br> V.019.D. The beginning teacher communicates mathematical ideas using a variety of representations (e.g., numeric, verbal, graphical, pictorial, symbolic, concrete). <br> Geometry TEKS <br> b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships. <br> b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic. <br> b.3.B. The student constructs and justifies statements about geometric figures and their properties. <br> b.3.E. The student uses deductive reasoning to prove a statement. <br> b.4. The student uses a variety of representations to describe geometric relationships and solve problems. <br> Background: Participants need to be familiar with triangle and quadrilateral properties. <br> Materials: compass, easel paper, colored markers, centimeter ruler <br> New Terms: vesica pisces <br> Procedures:

## Distribute the activity sheet.

Circles $A$ and $B$ pass through each others' centers and intersect at $C$ and $D$.


Draw the following segments:
$\overline{A B}, \overline{A C}, \overline{A D}, \overline{B C}, \overline{B D}, \overline{C D}$.
$\overleftrightarrow{A B}$ intersects circle $A$ at $P$, and circle $B$ at $Q$.
Draw $\overline{P Q}, \overline{P C}, \overline{P D}, \overline{Q C}, \overline{Q D}$.
$\overline{A B}$ intersects $\overline{C D}$ at $X$.

In your group identify and classify triangles and quadrilaterals. Explain why the properties you have identified are true. For example, if a rhombus is identified on the basis of four congruent sides, explain why the four sides are congruent.

Allow participants $30-45$ minutes to work together to complete the activity. Then ask individuals to share results, while one participant records on a sheet of easel paper.

This figure is called a vesica pisces. A vesica pisces is created by two identical intersecting circles, the circumference of one intersecting the center of the other.

All of the figures within the vesica pisces can be derived from the congruent radii $\overline{A B}, \overline{A C}, \overline{A D}, \overline{B C}$, and $\overline{B D}$. Some properties and figures follow. Many more can be discerned.

- ADBC is a rhombus. $(\overline{A C} \cong \overline{A D} \cong \overline{B C} \cong \overline{B D}$. $)$
- $\overline{A B} \perp \overline{C D}$. (Diagonals of a rhombus are perpendicular to each other.)
- $A X=X B$. (Diagonals of a rhombus bisect each other.) $P A=B Q$ (Radii of congruent circles are congruent.) Therefore, $P A+A X=P X=X B+B Q=X Q$.
- PCQD is a rhombus. (Diagonals of a rhombus are perpendicular bisectors of each other.)
- $\triangle A B C$ is equilateral. $(A B=A C=B C$.) Similarly, $\triangle A B D$ is equilateral.
- $m \angle P A C=120^{\circ} .\left(m \angle C A B=60^{\circ}\right.$, since $\triangle A B C$ is equilateral. $)$
- $m \angle A C D=30^{\circ}$. (Diagonal $\overline{C D}$ of rhombus $A C B D$ bisects $\angle A C B$.)
- $m \angle P C A=30^{\circ}$. $\left(\triangle P A C\right.$ is isosceles since $\left.P A=A C ; m \angle P A C=120^{\circ}.\right)$
- $m \angle P C D=60^{\circ} .(m \angle A C D+m \angle P C A=m \angle P C D$.
- $\triangle P C D$ is equilateral. ( $\angle P C D \cong \angle P D C$, since the diagonals lie on the line of symmetry.)
- $m \angle D A C=120^{\circ}=2 m \angle D P C$. (The measure of the arc is two times the measure of the inscribed angle intercepting the arc.)
Conclude the unit with the following discussion:

Explain to participants that the vesica pisces is the concept behind the traditional compass and straight edge construction of the perpendicular bisector of a segment, in this case,
$\overline{A B}$. On a clean sheet of paper participants draw a segment and construct the perpendicular bisector using a compass. They then draw congruent circles with centers at the ends of the segments and radii equal to the lengths of the segments.

Participants are performing at the van Hiele Relational Level because they use logical justifications to explore relationships among properties of quadrilaterals.

## Vesica Pisces

The vesica pisces on the lid of Chalice Well was
 designed by the excavator of Glastonbury Abbey, Frederick Bligh Bond, resident archaeologist of Glastonbury Abbey in the early 1900s. It was given to the Chalice Well as a thanks-offering for Peace in 1919, at the end of World War One, by friends and lovers of the Well and of Glastonbury.


Circles $A$ and $B$ pass through each others' centers and intersect at $C$ and $D$. Draw the following segments: $\overline{A B}, \overline{A C}, \overline{A D}, \overline{B C}, \overline{B D}, \overline{C D}$.
$\overleftrightarrow{A B}$ intersects circle $A$ at $P$ and circle $B$ at $Q$.
Draw $\overline{P Q}, \overline{P C}, \overline{P D}, \overline{Q C}, \overline{Q D}$.
$\overline{A B}$ intersects $\overline{C D}$ at $X$.
In your group identify and classify triangles and quadrilaterals using side lengths and angle measurements. Explain why the properties you have identified are true. For example, if a rhombus is identified on the basis of four congruent sides, explain why the four sides are congruent.

## Exploring Prisms

Overview: Participants construct prisms by using the polygon as the base and the translation vectors from its vertices to construct a prism. This is done in a three-dimensional coordinate system. Participants explore and describe attributes of prisms.

Objectives: TExES Mathematics Competencies
II.004.A. The beginning teacher recognizes and extends patterns and relationships in data presented tables, sequences, or graphs.
III.011.B. The beginning teacher applies formulas for perimeter, area, surface area, and volume of geometric figures and shapes (e.g., polygons, pyramids, prisms, cylinders).
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two-and three-dimensional figures and shapes (e.g., relationships of sides, angles).
III.014.D. The beginning teacher applies transformations in the coordinate plane.
V.019.A. The beginning teacher recognizes and uses multiple representations of a mathematical concept (e.g., a point and its coordinates, the area of a circle as a quadratic function of the radius, probability as the ratio of two areas, area of a plane region as a definite integral).
V.019.C. The beginning teacher translates mathematical ideas between verbal and symbolic forms.
V.019.D. The beginning teacher communicates mathematical ideas using a variety of representations (e.g., numeric, verbal, graphical, pictorial, symbolic, concrete).

## Geometry TEKS

b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.4. The student selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems. d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.
e.2.D. The student analyzes the characteristics of three-dimensional figures and their component parts.
e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.

Background: Participants should be able to describe and perform a coordinate translation in two dimensions.

Materials: cardstock, scissors, floral wire, modeling clay, one-inch grid easel paper, Flash animation video 3-D.html, computer lab and/or computer with projector

New Terms: prism

## Procedures:

Assign a different polygon to each group of participants. After translating the polygons on grid paper according to the rule $(x, y) \rightarrow(x, y+5)$, each group constructs a prism with the assigned polygon as its base. Suggestions follow:

Group 1: triangle
Group 2: square
Group 3: rectangle (non-square)
Group 4: regular pentagon
Group 5: regular hexagon
Group 6: regular octagon
Note: A template is provided for the regular pentagon, hexagon, and octagon in the activity pages.

1. On one-inch grid easel paper plot and label the coordinates of the vertices of your polygon. Connect the vertices to sketch the polygon.
In a two-dimensional coordinate plane, have participants plot ordered pairs that form the vertices of the assigned polygon. They then connect the points to sketch the polygon.
2. Translate the polygon in the $x y$-plane according to the rule $(x, y) \rightarrow(x, y+5)$. Draw the image and the pre-image on the same coordinate grid. Draw the translation vectors.

Have participants translate the polygon according to the rule $(x, y) \rightarrow(x, y+5)$. They should connect the vertices of the original polygon to the corresponding vertices of its image. A sample figure with translation vectors is shown on the next page.

3. On a piece of cardstock, construct and cut out two polygons that are congruent to the one you plotted on the grid paper. Cut equal lengths of floral wire to use to connect the corresponding vertices of the two polygons. Use small balls of modeling clay to attach the ends of the floral wire to the corresponding vertices of the two polygons. Place one cardstock polygon over the original polygon and the other polygon over the image of the original polygon.

Note: Participants will need to keep their wire-frame prisms for use again in Exploring Pyramids and Cones.
4. What do the pieces of floral wire represent?

The pieces of floral wire represent the translation vectors. When the translation vectors are coplanar with the polygon, all parts of the figure are in the plane, in this case the xy-plane.
5. Keeping the floral wire connected to the polygons, translate the image out of the plane.
A sample translation is shown on the next page.
Use the Flash animation video 3-D.html to demonstrate the translation.
$\mathrm{T}(x, y) \rightarrow(x, y+5)$
$0^{\circ}$ in the plane

$\mathrm{T}(x, y, z) \rightarrow(x, y, z+5)$
$90^{\circ}$ out of the plane

6. When the translation vectors no longer lie in the same plane as the two polygons, describe the resulting figure.
When the translation vectors are not in the xy-plane, the figure becomes a three-dimensional solid, a prism, generated by translating the polygon along these vectors. The original polygon and its image then become the bases of the prism and lie in parallel planes. The figures bounded by the florist wire and the edges of the bases are rectangles.

Remind participants to add the new term prism to their glossaries.
7. Complete the table below for your wire frame figure.
(Sample answer is shown for a square prism.)

| Number of <br> faces | Number of <br> edges | Number of <br> vertices | Height |
| :---: | :---: | :---: | :---: |
| 6 | 12 | 8 | 5 in |

8. Collect data from the other groups to complete the table below.

Sample data are shown in the table.

| Name of Polygon | Number of faces | Number of <br> edges | Number of <br> vertices |
| :---: | :---: | :---: | :---: |
| Triangle | 5 | 9 | 6 |
| Quadrilateral | 6 | 12 | 8 |
| Pentagon | 7 | 15 | 10 |
| Hexagon | 8 | 18 | 12 |
| Octagon | 10 | 24 | 16 |
| Decagon | 12 | 30 | 20 |

9. What patterns and relationships do you notice?

Answers will vary but may include the following:

- There are more edges than faces
- The number of vertices on the prism is twice the number of vertices on the original polygon
- The number of edges on the prism is three times the number of edges on the original polygon
- The number of faces on the prism is equal to the number of edges on the original polygon plus two

10 . Use your table to discover a rule relating the number of faces, $F$, number of edges, $E$, and number of vertices, $V$, of a prism.
Answers will vary but may include:

- $\quad V+F=E+2$
- $V+F-2=E$
- $\quad V-2=E-F$
- $V+F-E=2$

This relationship, $V+F-E=2$, describes the Euler characteristic or Euler number for convex polyhedra.

Participants are working at the Descriptive Level in describing the properties of a prism. They are using inductive reasoning throughout this activity.

## Exploring Prisms

Your instructor will assign a polygon to your group. Record the name of your polygon below.

## Polygon:

1. On one-inch grid easel paper plot and label the coordinates of the vertices of your polygon. Connect the vertices to sketch the polygon.
2. Translate your polygon in the $x y$-plane according to the rule $(x, y) \rightarrow(x, y+5)$. Draw the image and pre-image on the same coordinate grid. Draw the translation vectors.
3. On a piece of cardstock, construct and cut out two polygons that are congruent to the one you plotted on the grid paper. Cut equal lengths of floral wire to use to connect the corresponding vertices of your two polygons. Use small balls of modeling clay to attach the ends of the floral wire to the corresponding vertices of the two polygons. Place one cardstock polygon over the original polygon and the other polygon over the image of the original polygon.
4. What do the pieces of floral wire represent?
5. Keeping the floral wire connected to the polygons, translate the image out of the plane.
6. When the translation vectors no longer lie in the same plane as the two polygons, describe the resulting figure.
7. Complete the table below for your wire frame figure.

| Number of <br> faces | Number of <br> edges | Number <br> of vertices | Height |
| :---: | :---: | :---: | :---: |
|  |  |  |  |

8. Collect data from the other groups to complete the table below.

| Name of <br> Polygon | Number of <br> faces | Number of <br> edges | Number of <br> vertices |
| :--- | :--- | :--- | :--- |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
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|  |  |  |  |

9. What patterns and relationships do you notice?
10. Use your table to discover a rule relating the number of faces, $F$, the number of edges, $E$, and the number of vertices, $V$, of a prism.

## Template - Regular Pentagon



## Template - Regular Hexagon



Template - Regular Octagon


## References and Additional Resources

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# Unit 4 - Informal Logic/Deductive Reasoning 

## Informal Language

| Overview: | Participants learn/review some of the language and notation used in <br> informal logic. |
| :--- | :--- |
| Objective: | TExES Mathematics Competencies <br> III.012.A. The beginning teacher understands axiomatic systems and <br> their components (e.g., undefined terms, defined terms, theorems, <br> examples, counterexamples). |
|  | V.018.D. The beginning teacher uses formal and informal reasoning to |
| justify mathematical ideas. |  |

## Geometry TEKS

b.1.A. The student develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems.
b.3.A. The student determines if the converse of a conditional statement is true or false.

Background: Presenter will provide background information for participants.
Materials: easel paper, colored markers
New terms: argument, biconditional statement, conditional statement, contrapositive statement, converse statement, inverse statement

## Procedures:

This activity introduces terms and mathematical notation that will be used later in this unit. Before participants begin work on the activity, define and apply terms with the example:

A logical argument consists of a set of premises and a conclusion. Example: "Mr. French is the only calculus teacher. Mr. French, Ms. Anderson, Ms. Allen, and Ms. Short teach pre-calculus."

When you write "If I take calculus, then Mr. French is my teacher" you are writing a conditional statement.
"I take calculus" is the premise and "Mr. French is my teacher" is the conclusion. To create the converse of a conditional statement, the two parts of the conditional statement are simply interchanged.
"If Mr. French is my teacher, then I take calculus."
Is the converse of a conditional statement always true when the conditional statement is true?
Not always, because, in our example, I could take pre-calculus from Mr. French.
The negation of a sentence is made by placing the word not in the sentence appropriately. To create the inverse, the two parts of a conditional are negated.
"If I do not take calculus, then Mr. French is not my teacher."
Is the inverse of a conditional statement always true when the conditional statement is true?
Not always, because, in our example, Mr. French could be my pre-calculus teacher.
To create the contrapositive, the two parts of the conditional are reversed and negated. "If Mr. French is not my teacher, then I don't take calculus."

Is the contrapositive of a conditional statement always true when the conditional statement is true?
Yes, it is true.
Remind participants to add the terms argument, conditional statement, converse statement, inverse statement, and contrapositive statement to their glossaries.

Have each group work on the activity page. Provide each group with easel paper and markers. After groups complete the activity page, have each group present a different problem to the entire group.

Write the given sentences (1-4) as conditional statements. Then find their converses, inverses, and contrapositives. Assuming the conditional statements are true, determine whether each of the converse, inverse, and contrapositive statements is true or false. Give an explanation for each false statement.

1. I use an umbrella when it rains.

Conditional: If it rains, then I use an umbrella.
Converse: If I use an umbrella, then it rains.
Inverse: If it does not rain, then I do not use an umbrella.
Contrapositive: If I do not use an umbrella, then it does not rain.
The converse is not always true, since I may also use my umbrella on a very sunny day.
The inverse is not always true, because if it does not rain, it may be a very sunny day and I may use my umbrella.
The contrapositive is true.
2. A rhombus is a quadrilateral with four congruent sides.

Conditional: If a quadrilateral is a rhombus, then it has four congruent sides.

Converse: If a quadrilateral has four congruent sides, then it is a rhombus.
Inverse: If a quadrilateral is not a rhombus, then it does not have four congruent sides. Contrapositive: If a quadrilateral does not have four congruent sides, then it is not a rhombus.
The conditional is true. The converse, the inverse, and the contrapositive are true.
3. The sum of the measures of the interior angles of a triangle is $180^{\circ}$.

Conditional: If a polygon is a triangle, then the sum of the measures of the interior angles is $180^{\circ}$.
Converse: If the sum of the measures of the interior angles is $180^{\circ}$, then the polygon is a triangle.
Inverse: If a polygon is not a triangle, then the sum of the measures of the interior angles is not $180^{\circ}$.
Contrapositive: If the sum of the measures of the interior angles is not $180^{\circ}$, then the polygon is not a triangle.
The conditional is true. The converse, the inverse, and the contrapositive are true.
4. Vertical angles are congruent.

Conditional: If two angles are vertical angles, then they are congruent.
Converse: If two angles are congruent, then they are vertical angles.
Inverse: If two angles are not vertical angles, then they are not congruent.
Contrapositive: If two angles are not congruent, then they are not vertical angles.
The conditional is true. The converse is not always true If an angle is bisected, then the two smaller angles are congruent but not vertical angles.
The inverse is not always true. If two right angles are adjacent, then they are not vertical, but they are congruent.
The contrapositive is true.
5. Write a real-world example of a conditional statement with a true converse.

Possible Answer:
My cat and dog always eat together.
Conditional: If my cat eats, then my dog eats.
Converse: If my dog eats, then my cat eats.
6. Write a real-world example of a conditional statement with a false converse.

Possible Answer:
Every Mathlete at Lanier Middle School is an $8^{\text {th }}$-grade student.
Conditional: If a Lanier Middle School student is a Mathlete, then he/she is an $8^{\text {th }}$ -
grade student.
Converse: If a Lanier Middle School student is an $8^{\text {th }}$-grade student, then he/she is a Mathlete.
We do not know that every $8^{\text {th }}$-grade student is a Mathlete.
7. What conclusions can be made about the truth of the converse, inverse, and contrapositive statements for a given conditional that is true?
When two statements are either both true or both false they form a biconditional
statement. A conditional and its contrapositive form a biconditional statement. The converse and inverse statements of a conditional statement also form a biconditional statement.

Remind participants to add the term biconditional statement to their glossaries.
Close the activity with a discussion of the van Hiele levels for this activity. Success in this activity indicates that participants are working at the Relational Level or approaching the Deductive Level, because they informally recognize relationships among a conditional statement and its contrapositive, converse, and inverse statements.

## Informal Language

Write the given sentences (1-4) as conditional statements then find their converses, inverses, and contrapositives. Assuming the conditional statements are true, determine whether each of the converse, inverse, and contrapositive statements is true or false. Give an explanation for each false statement.

1. I use an umbrella when it rains.
2. A rhombus is a quadrilateral with four congruent sides.
3. The sum of the measures of the interior angles of a triangle is $180^{\circ}$.
4. Vertical angles are congruent.
5. Write a real-world example of a conditional statement with a true converse.
6. Write a real-world example of a conditional statement with a false converse.
7. What conclusions can be made about the truth of converse, inverse, and contrapositive statements when the conditional is true?

## Inductive Triangle Congruence

Overview: This activity develops the triangle congruence theorems using an inductive approach.

## Objective: TExES Mathematics Competencies

III.012.A. The beginning teacher understands axiomatic systems and their components (e.g., undefined terms, defined terms, theorems, examples, counterexamples). III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.012.E. The beginning teacher describes and justifies geometric constructions made using compass and straight edge, reflection devices, and other appropriate technologies. III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
V.018.A. The beginning teacher understands the nature of proof, including indirect proof, in mathematics.
V.018.B. The beginning teacher applies correct mathematical reasoning to derive valid conclusions from a set of premises. V.018.C. The beginning teacher uses inductive reasoning to make conjectures and uses deductive methods to evaluate the validity of conjectures.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

## Geometry TEKS

b.1.A. The student develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems.
b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.D. The student uses inductive reasoning to formulate a conjecture.
b.3.E. The student uses deductive reasoning to prove a statement.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.
e.3.B. The student justifies and applies triangle congruence relationships.

Background: Participants should be familiar with congruent triangle properties.
Materials: unlined 8.5 in . by 11 in. paper, compass, centimeter ruler, protractor, spaghetti, scissors

New terms: deductive reasoning, inductive reasoning

## Procedures:

Begin by explaining the differences between inductive and deductive reasoning.
Inductive reasoning is the process of observing data, recognizing patterns, and making generalizations from your observations. Much of geometry uses inductive reasoning especially at the lower van Hiele levels. Discovering that the sum of the measures of the interior angles of a triangle is $180^{\circ}$ by tearing the angles of different triangles and observing that the sum of their measures is $180^{\circ}$ for each of the triangles is an example of inductive reasoning. Inductive reasoning is used in geometry to discovery properties of figures.

Deductive reasoning is the process of proving or demonstrating that if certain statements are accepted as true, then other statements can be shown to follow. When a lawyer uses evidence (premises) to prove his/her case (conclusion), he/she is using deductive reasoning. Deductive reasoning is used in geometry to draw conclusions from given information. Deductive reasoning is generally used at the higher van Hiele levels.

Remind participants to add the terms inductive reasoning and deductive reasoning to their glossaries.

We draw conclusions about the congruence relationship of two triangles using both types of reasoning. This activity uses an inductive approach (with examples taken from
Discovering Geometry: An Investigative Approach, $3^{\text {rd }}$ Edition, © 2003, pp. 100, 219, 220, 221, 225, and 226, with permission from Key Curriculum Press).

The activity will address the question "When are two triangles congruent?" Participants will complete the activity in groups. Have each member within a group complete all the constructions. Groups will need white paper, compasses, rulers, protractors, and/or spaghetti to complete the activity. Then lead a whole-group discussion to formalize the congruence theorems.

Follow the directions to discover the circumstances under which two triangles are congruent. Figures may be constructed using compass, ruler, protractor, or spaghetti.

1. Construct a triangle on paper from the three measurements given. Cut strips of paper to the appropriate lengths or use spaghetti cut to the appropriate lengths. Be sure you match up the endpoints labeled with the same letter.
$A C=4 \mathrm{in}$.
$B C=5$ in.
$A B=7$ in.
Compare your triangle with the triangles made by others in your group. (One way to compare them is to place the triangles on top of each other to see if they coincide.) Is it possible to construct different triangles from the given information or will all the triangles be congruent?

Remind participants that they cannot arbitrarily select any three lengths for the sides of a triangle to be assured that those sides will form a triangle.

Side-Side-Side (SSS):
If the three sides of one triangle are congruent to the three sides of another triangle, what can we conclude?
The triangles are congruent. This is known as the Side-Side-Side Triangle Congruence Theorem (SSS).
2. Construct a triangle from the measurements given. Be sure to match up the endpoints labeled with the same letter.
$D E=6 \mathrm{in}$.
$D F=5$ in.
$\mathrm{m} \angle D=20^{\circ}$
Compare your triangle with the triangles made by others in your group. (One way to compare them is to place the triangles on top of each other to see if they coincide.) Is it possible to construct different triangles from the given information or will all the triangles be congruent?

Side-Angle-Side (SAS):
If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, what can we conclude?
The triangles are congruent. This is known as the Side-Angle-Side Triangle
Congruence Theorem (SAS).
3. Construct a triangle from the three measurements given. Be sure that the side is included between the given angles.
$M T=8 \mathrm{in}$.
$\mathrm{m} \angle M=30^{\circ}$
$\mathrm{m} \angle T=50^{\circ}$

Compare your triangle with the triangles made by others in your group. Is it possible to construct different triangles from the given information or will all the triangles be congruent?

Angle-Side-Angle (ASA):
If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, what can we conclude?
The triangles are congruent. This is known as the Angle-Side-Angle Triangle Congruence Theorem (ASA).
4. Construct a triangle from the three measurements given.
$S T=6$ in.
$T U=3 \mathrm{in}$.
$\mathrm{m} \angle S=20^{\circ}$
Compare your triangle with the triangles made by others in your group. Is it possible to construct different triangles from the given information or will all the triangles be congruent?

Side-Side-Angle (SSA):
If two sides and a non-included angle of one triangle are congruent to two sides and a non-included angle of another triangle, what can we conclude?
The triangles are not always congruent, because two different triangles are possible.
Therefore, there is no Side-Side-Angle congruence theorem.
5. Construct a triangle from the three measurements given.
$\mathrm{m} \angle M=50^{\circ}$
$\mathrm{m} \angle N=60^{\circ}$
$\mathrm{m} \angle O=70^{\circ}$

Compare your triangle with the triangles made by others in your group. (One way to compare them is to place the triangles on top of each other to see if they coincide.) Is it possible to construct different triangles from the given information or will all the triangles be congruent?

Angle-Angle-Angle (AAA):
If three angles of one triangle are congruent to three angles of another, what can we conclude?
The triangles are not always congruent, because there are an infinite number of possible triangles of different side lengths with those angle measures. Therefore, there is no Angle-Angle-Angle congruence theorem.
6. In $\triangle A B C$ and $\triangle X Y Z$ given below, label $\angle A \cong \angle X, \angle B \cong \angle Y$, and $\overline{B C} \cong \overline{Y Z}$. Is $\triangle A B C \cong \triangle X Y Z$ ? Explain your answer.


If two angles in one triangle are congruent to two angles in another, then the third pair of angles are congruent, i.e. $\angle C \cong \angle Z$. So we now have two angles and the included side of one triangle congruent to two angles and the included side of another.
By the ASA Congruence Theorem, $\triangle A B C \cong \triangle X Y Z$. The AAS Congruence
Theorem follows directly from the ASA Congruence Theorem.
Angle-Angle-Side Triangle Congruence Theorem (AAS):
If two angles and a non-included side of one triangle are congruent to the corresponding angles and side of another triangle, what can we conclude? The triangles are congruent. This is known as the Angle-Angle-Side Triangle Congruence Theorem (AAS).

Success with this activity indicates that participants are working initially at the Descriptive Level, as they use inductive methods to determine triangle congruence. Participants approach the Deductive Level in 6, when they relate properties from previously determined combinations of properties.

## Inductive Triangle Congruence

Follow the directions to discover the circumstances under which two triangles are congruent. Figures may be constructed using compass, ruler, protractor, or spaghetti.

1. Construct a triangle on paper from the three measurements given. Cut strips of paper to the appropriate lengths or use spaghetti cut to the appropriate lengths. Be sure you match up the endpoints labeled with the same letter.
$A C=4 \mathrm{in}$.
$B C=5$ in.
$A B=7$ in.
Compare your triangle with the triangles made by others in your group. (One way to compare them is to place the triangles on top of each other to see if they coincide.) Is it possible to construct different triangles from the given information or will all the triangles be congruent?

Side-Side-Side (SSS):
If the three sides of one triangle are congruent to the three sides of another triangle, what can we conclude?
2. Construct a triangle from the measurements given. Be sure to match up the endpoints labeled with the same letter.
$D E=6$ in.
$D F=5$ in.
$\mathrm{m} \angle D=20^{\circ}$
Compare your triangle with the triangles made by others in your group. (One way to compare them is to place the triangles on top of each other to see if they coincide.) Is it possible to construct different triangles from the given information or will all the triangles be congruent?

Side-Angle-Side (SAS):

If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, what can you conclude?
3. Construct a triangle from the three measurements given. Be sure that the side is included between the given angles.
$M T=8 \mathrm{in}$.
$\mathrm{m} \angle M=30^{\circ}$
$\mathrm{m} \angle T=50^{\circ}$
Compare your triangle with the triangles made by others in your group. Is it possible to construct different triangles from the given information or will all the triangles be congruent?

Angle-Side-Angle (ASA):
If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, what can you conclude?
4. Construct a triangle from the three measurements given.
$S T=6$ in.
$T U=3$ in.
$\mathrm{m} \angle S=20^{\circ}$
Compare your triangle with the triangles made by others in your group. Is it possible to construct different triangles from the given information, or will all the triangles be congruent?

Side-Side-Angle (SSA):
If two sides and a non-included angle of one triangle are congruent to two sides and a non-included angle of another triangle, what can we conclude?
5. Construct a triangle from the three measurements given.
$\mathrm{m} \angle M=50^{\circ}$
$\mathrm{m} \angle N=60^{\circ}$
$\mathrm{m} \angle O=70^{\circ}$
Compare your triangle with the triangles made by others in your group. (One way to compare them is to place the triangles on top of each other to see if they coincide.) Is it possible to construct different triangles from the given information or will all the triangles be congruent?

Angle-Angle-Angle (AAA):
If three angles of one triangle are congruent to three angles of another, what can you conclude?
6. In $\triangle A B C$ and $\triangle X Y Z$ given below, label $\angle A \cong \angle X, \angle B \cong \angle Y$, and $\overline{B C} \cong \overline{Y Z}$. Is $\triangle A B C \cong \triangle X Y Z$ ? Explain your answer.


Angle-Angle-Side (AAS):
If two angles and a non-included side of one triangle are congruent to the corresponding angles and side of another triangle, what can we conclude?

## Deductive Triangle Congruence

Overview: Participants use the triangle congruence theorems to prove that given triangles are congruent.

Objective: TExES Mathematics Competencies<br>III.012.A. The beginning teacher understands axiomatic systems and their components (e.g., undefined terms, defined terms, theorems, examples, counterexamples).<br>III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.<br>V.018.A. The beginning teacher understands the nature of proof, including indirect proof, in mathematics.<br>V.018.B. The beginning teacher applies correct mathematical reasoning to derive valid conclusions from a set of premises. V.018.C. The beginning teacher uses inductive reasoning to make conjectures and uses deductive methods to evaluate the validity of conjectures.<br>V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

## Geometry TEKS

b.1.A. The student develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems.
b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.C. The student demonstrates what it means to prove mathematically that statements are true.
b.3.E. The student uses deductive reasoning to prove a statement.
e.3.B. The student justifies and applies triangle congruence relationships.

Background: Participants need to know the SSS, SAS, AAS, and ASA triangle congruence theorems.

## Materials:

New Terms: reflexive property

## Procedures:

If necessary, review the SSS, SAS, ASA, and AAS triangle congruence theorems. In the activity, participants will determine if two triangles are congruent, and if they are congruent, they will deductively prove congruence.

Participants will need to use the reflexive property in this activity. The reflexive property states that a number is equal to itself. It will be used to describe when two figures share a common side.
For example, in $\triangle D E G$ and $\triangle E F G$ shown below, $\overline{E G} \cong \overline{E G}$ by the reflexive property.


Remind participants to add the term reflexive property to their glossaries.
The following problems are taken from Discovering Geometry: An Investigative
Approach, $3^{\text {rd }}$ Edition, © 2003, pp. 227-228, 231 with permission from Key Curriculum Press.

Discuss the activity sheet with participants. List all the facts that may help prove that the two triangles in 1 are congruent. Participants work together to complete 2-14.

Determine if each pair of triangles below is congruent. List facts about the triangles that help in your determination and mark them in the figures. If they are congruent, state the congruence theorem used to prove the two triangles congruent.

2.

$\angle A \cong \angle D \quad$ (Given)
$\overline{A C} \cong \overline{C D}$
(Given)
$\angle A C B \cong \angle D C E$ (Vertical angles
are congruent.)
$\triangle A C B \cong \triangle D C E(A S A)$
3.


$$
\overline{A S} \cong \overline{L O}
$$

(Given)
(Given)
(Given)
$\Delta A G S \cong \triangle O I L \quad(A A S)$
4.

$\overline{H W} \cong \overline{F W}$
(Given)
$\angle H W O \cong \angle E W F \quad \begin{gathered}\text { (Vertical angles are } \\ \text { congruent.) }\end{gathered}$ congruent.)

Not enough information is given to prove that the two triangles are congruent.
5.

$\angle H F S \cong \angle S F I \quad$ (Given)
$\overline{H S} \cong \overline{S I}$
(Given)
$\overline{F S} \cong \overline{F S}$
(Reflexive property)

Not enough information is given to prove that the two triangles are congruent.
7.

$\overline{A D} \cong \overline{E D}$,
(Given)
$\overline{A F} \cong \overline{E F}$
$\overline{D F} \cong \overline{D F} \quad$ (Reflexive property)
$\triangle F A D \cong \triangle F E D(S S S)$
6.

$\overline{I N} \square \overline{A L}$
(Given)
$\angle I T N \cong \angle A T L$
(Vertical angles are congruent.)
$\angle T A L \cong \angle T I N$
$\angle T N I \cong \angle T L A$
(If two parallel lines are cut by a transversal, then the alternate interior angles are congruent.)

Not enough information is given to prove that the two triangles are congruent.
8.

$\overline{O H} \square \overline{A T}$,
(Given)
$\overline{H W} \cong \overline{W T}$
$\angle W O H \cong \angle W A T, \quad$ (If two parallel lines $\angle W H O \cong \angle W T A \quad$ are cut by a transversal, then the alternate interior angles are congruent.)
$\Delta W H O \cong \triangle W T A$
(ASA)
9.


| $\angle L A T \cong \angle T A S$ |  |
| :--- | :--- |
| $\overline{L A} \cong \overline{A S}$ |  |
| $\overline{A T} \cong \overline{A T}$ |  |
| Given) |  |
|  |  |
| (Reflexive property) |  |

$\Delta L A T \cong \triangle S A T$
10.

$m \angle F M R=90^{\circ} \quad$ (Given)
$m \angle A R M=90^{\circ} \quad$ (Given)
$\overline{F M} \cong \overline{A R} \quad$ (Given)
$\overline{M R} \cong \overline{M R} \quad$ (Reflexive property)
$\Delta R M F \cong \triangle M R A \quad(A A S)$
Use the triangle congruence theorems to answer the questions below. Explain your answers.


Given: $\overline{C N} \cong \overline{W N}$

$$
\angle C \cong \angle W
$$

Is $\overline{R N} \cong \overline{O N}$ ?

Yes
$\angle R N C \cong \angle O N W$
$\Delta C N R \cong \triangle W N O$
$\overline{R N} \cong \overline{O N}$
(Vertical angles are congruent.)

Is $\overline{C R} \cong \overline{H S}$ ?
12.


Given: $\overline{C S} \cong \overline{H R}$

$$
\angle 1 \cong \angle 2
$$

$$
0
$$

This cannot be determined. The congruent parts lead to the ambiguous case SSA for $\triangle S C H$ and $\triangle R H C$.

## congruent.)

13. 



Given: $\angle S \cong \angle I$
$\angle G \cong \angle A$
$T$ is the midpoint of $\overline{S I}$
Is $\overline{S G} \cong \overline{I A}$ ?

Yes
$\overline{T S} \cong \overline{I T}$
(Definition of midpoint)
$\Delta T S G \cong \Delta T I A$
$\overline{S G} \cong \overline{I A}$
(Corresponding parts
of congruent triangles are congruent.)
14.


Given: $H A L F$ is a parallelogram.
Is $\overline{H A} \cong \overline{H F}$ ?

This cannot be determined.
$\Delta H L F \cong \triangle L H A$ by ASA. However, $\overline{H A}$ and $\overline{H F}$ are not corresponding sides of congruent triangles, therefore we know nothing about the relationship of their lengths.

Success with this activity indicates that participants are working at the Deductive Level, because they produce formal deductive arguments.

## Deductive Triangle Congruence

Determine if each pair of triangles below is congruent. List facts about the triangles that help in your determination and mark them in the figures. If they are congruent, state the congruence theorem used to prove the two triangles congruent.


Given: $\angle F \cong \angle A$

$$
\begin{aligned}
& \angle D \cong \angle C \\
& \overline{D F} \cong \overline{A C}
\end{aligned}
$$


4.


Given: $\overline{H W} \cong \overline{F W}$
5.


Given: $\angle H F S \cong \angle S F I$ $\overline{H S} \cong \overline{S I}$
6.


Given: $\overline{I N} \square \overline{A L}$
8.


Given: $\overline{O H} \square \overline{A T}$
$\overline{H W} \cong \overline{W T}$
10.


Given: $\mathrm{m} \angle F M R=90^{\circ}$

$$
\overline{L A} \cong \overline{A S} \quad \begin{aligned}
& \mathrm{m} \angle A R M=90^{\circ} \\
& \overline{F M} \cong \overline{A R}
\end{aligned}
$$

Use the triangle congruence theorems to answer the questions below. Explain your answers.
11.


Given: $\overline{C S} \cong \overline{H R}$
$\angle C \cong \angle W$
Is $\overline{R N} \cong \overline{O N}$ ?
Is $\overline{C R} \cong \overline{H S}$ ?
13.


Given: $\angle S \cong \angle I$
$\angle G \cong \angle A$
T is the midpoint of $\overline{S I}$
Is $\overline{S G} \cong \overline{I A}$ ?
14.


Given: HALF is a parallelogram.
Is $\overline{H A} \cong \overline{H F}$ ?

## Quadrilateral Proofs

Overview: Using definitions of quadrilaterals and triangle congruence theorems, participants prove properties of quadrilaterals.

## Objective: TExES Mathematics Competencies

III.012.A. The beginning teacher understands axiomatic systems and their components (e.g., undefined terms, defined terms, theorems, examples, counterexamples).
III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
V.018.A. The beginning teacher understands the nature of proof, including indirect proof, in mathematics.
V.018.B. The beginning teacher applies correct mathematical reasoning to derive valid conclusions from a set of premises. V.018.C. The beginning teacher uses inductive reasoning to make conjectures and uses deductive methods to evaluate the validity of conjectures.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

## Geometry TEKS

b.1.A. The student develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning and theorems.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.C. The student demonstrates what it means to prove mathematically that statements are true.
b.3.E. The student uses deductive reasoning to prove a statement.
e.3.B. The student justifies and applies triangle congruence relationships.

Background: Participants need a knowledge of definitions and rules learned in previous units.

Materials: easel paper, colored markers

## New Terms:

## Procedures:

Using the congruence theorems and definitions presented earlier in this module, participants will prove theorems about quadrilaterals. These activities are taken from

Discovering Geometry: An Investigative Approach, $3^{\text {rd }}$ Edition, © 2003, pp. 692-693 with permission from Key Curriculum Press.

Lead a discussion to prove that a diagonal of a parallelogram divides the parallelogram into two congruent triangles.

Put the following on a transparency or on easel paper.


Given: Parallelogram $A B C D$ with diagonal $\overline{A C}$
Prove: $\triangle A B C \cong \triangle C D A$

Have participants list facts that they observe from the figure.

Quadrilateral $A B C D$ is a parallelogram

$$
\begin{aligned}
& \overline{A B} \| \overline{D C} \text { and } \overline{A D} \| \overline{B C} \\
& \angle C A B \cong \angle A C D \text { and } \angle B C A \cong \angle D A C
\end{aligned}
$$

$$
\overline{A C} \cong \overline{A C}
$$

$$
\triangle A B C \cong \triangle C D A
$$

(Given)
(Opposite sides of a parallelogram are parallel.)
(If two parallel lines are cut by transversal, then the alternate interior angles are congruent.)
(Reflexive property)
(ASA)

This proves that a diagonal of a parallelogram divides the parallelogram into two congruent triangles.

Participants should work in groups to prove $1-6$ on the activity sheet. They may use the above theorem in their proofs. Give each group easel paper and markers. When all the groups are finished, have each group present on easel paper a different proof to the entire class.

Work with participants in your group to prove the statements below. Before you try to prove each statement, draw a diagram and state both what is given and what you are proving in terms of your diagram.

1. Prove that the opposite sides of a parallelogram are congruent.


Given: Parallelogram $A B C D$ with diagonal $\overline{A C}$
Prove: $\overline{A B} \cong \overline{D C}, \overline{A D} \cong \overline{B C}$

Parallelogram $A B C D$ with diagonal $\overline{A C}$
$\triangle A B C \cong \triangle C D A$
$\overline{A B} \cong \overline{D C}$ and $\overline{A D} \cong \overline{B C}$
(Given)
(A diagonal of a parallelogram divides the parallelogram into two congruent triangles.)
(Corresponding parts of congruent triangles are congruent.)
2. Prove that the opposite angles of a parallelogram are congruent.


Given: Parallelogram $A B C D$ with diagonal $\overline{A C}$
Prove: $\angle D \cong \angle B, \angle D A B \cong \angle B C D$

Parallelogram $A B C D$ with diagonal $\overline{A C}$
$\triangle A B C \cong \triangle C D A$
$\angle D \cong \angle B$
(Given)
(A diagonal of a parallelogram divides the parallelogram into two congruent triangles.)
(Corresponding parts of congruent triangles are congruent.)

Similarly if we use $\overline{B D}$ as the diagonal instead of $\overline{A C}$, then the congruent angles would be $\angle D A B$ and $\angle B C D$.
3. State and prove the converse of 1 above.

If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.


Given: Quadrilateral $A B C D, \overline{A B} \cong \overline{D C}, \overline{A D} \cong \overline{B C}$
Prove: Quadrilateral $A B C D$ is a parallelogram

Quadrilateral $A B C D$ with a diagonal $\overline{A C}$
$\overline{A B} \cong \overline{D C}, \overline{A D} \cong \overline{B C}$
$\overline{A C} \cong \overline{A C}$
$\triangle A D C \cong \triangle C B A$
$\angle D C A \cong \angle B A C$ and $\angle D A C \cong \angle A C B$
$\overline{D C} \square \overline{A B}$ and $\overline{D A} \square \overline{B C}$
(Given)
(Given)
(Reflexive property)
(SSS)
(Corresponding parts of congruent triangles are congruent.)
(If two lines are cut by a transversal so that the alternating angles are congruent, then the two lines are parallel.)
(Definition of parallelogram)

Quadrilateral $A B C D$ is a parallelogram
A parallel proof could have been constructed using $\overline{B D}$ instead of $\overline{A C}$ as the diagonal of the quadrilateral to prove $\triangle B A D \cong \triangle D C B$.
4. Prove that each diagonal of a rhombus bisects the two opposite angles.


Given: Rhombus $A B C D$ with diagonal $\overline{A C}$

Prove: $\angle D A C \cong \angle B A C$ and $\angle D C A \cong \angle B C A$

Rhombus $A B C D$ with diagonal $\overline{A C}$
$\overline{A B} \cong \overline{B C} \cong \overline{C D} \cong \overline{D A}$
$\overline{A C} \cong \overline{A C}$
$\triangle A B C \cong \triangle A D C$
$\angle D A C \cong \angle B A C$ and $\angle D C A \cong \angle B C A$
(Given)
(Definition of rhombus)
(Reflexive property)
(SSS)
(Corresponding parts of congruent triangles are congruent.)
$\angle D A B$ and $\angle D C B$ are bisected by $\overline{A C}$
(Definition of angle bisector)
A parallel proof could have been constructed using $\overline{B D}$ instead of $\overline{A C}$ as the diagonal of the rhombus to prove that $\overline{B D}$ bisects $\angle D C B$ and $\angle D A B$.
5. Prove that the diagonals of a rectangle are congruent.


Given: Rectangle $A B C D$ with diagonals $\overline{A C}$ and $\overline{B D}$
Prove: $\overline{A C} \cong \overline{B D}$
$\angle D A B$ and $\angle C B A$ are right angles
(Definition of rectangle)
$\angle A D C$ and $\angle B C D$ are right angles
$\angle D A B \cong \angle C B A$ and $\angle A D C \cong \angle B C D$
$\angle D A B$ and $\angle C B A$ are supplementary angles
(All right angles are congruent.)
$\angle A D C$ and $\angle B C D$ are supplementary angles
$\overline{D A} \square \overline{B C}$ and $\overline{A B} \square \overline{D C}$
$A B C D$ is a parallelogram
(Definition of supplementary angles)
(If two lines are cut by a transversal so that the interior angles on the same side of the transversal are supplementary, then the lines are parallel.)
(Definition of a parallelogram)

$$
\begin{array}{ll}
\overline{A B} \cong \overline{C D} \text { and } \overline{A D} \cong \overline{B C} & \begin{array}{l}
\text { (Opposite sides of a parallelogram } \\
\text { are congruent.) }
\end{array} \\
\overline{A B} \cong \overline{A B} & \text { (Reflexive property) } \\
\Delta D A B \cong \triangle C B A & (S S S) \\
\overline{A C} \cong \overline{B D} & \begin{array}{l}
\text { (Corresponding sides of congruent } \\
\text { triangles are congruent.) }
\end{array}
\end{array}
$$

6. Prove that the angles between each pair of congruent sides of a kite are bisected by a diagonal.


Given: Kite $A B C D$ with diagonals $\overline{A C}$ and $\overline{B D}$
Prove: $\angle A B D \cong \angle C B D$ and $\angle A D B \cong \angle C D B$

Kite $A B C D$ with diagonals $\overline{A C}$ and $\overline{B D}$
$\overline{A B} \cong \overline{B C}$ and $\overline{A D} \cong \overline{C D}$
$\overline{B D} \cong \overline{B D}$
$\triangle A B C \cong \triangle C B D$
$\angle A B D \cong \angle C B D$ and $\angle A D B \cong \angle C D B$
(Given)
(Definition of a kite)
(Reflexive property)
(Corresponding parts of congruent triangles are congruent.)

Success with this activity indicates that participants are working at the Deductive Level because they develop formal deductive proofs.

## Quadrilateral Proofs

Work with participants in your group to prove the statements below. Before you try to prove each statement, draw a diagram, state what is given and what you are proving in terms of your diagram.

1. Prove that the opposite sides of a parallelogram are congruent.
2. Prove that the opposite angles of a parallelogram are congruent.
3. State and prove the converse of 1 above.
4. Prove that each diagonal of a rhombus bisects the two opposite angles.
5. Prove that the diagonals of a rectangle are congruent.
6. Prove that the angles between each pair of congruent sides of a kite are bisected by a diagonal.

## Alternate Definitions of Quadrilaterals

Overview: Participants write alternative definitions of quadrilaterals based on their properties.

## Objective: TExES Mathematics Competencies

III.012.A. The beginning teacher understands axiomatic systems and their components (e.g., undefined terms, defined terms, theorems, examples, counterexamples).
III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.013.A. The beginning teacher analyzes the properties of polygons and their components.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
V.018.B. The beginning teacher applies correct mathematical reasoning to derive valid conclusions from a set of premises.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

## Geometry TEKS

b.1.A. The student develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning and theorems.
b.3.A. The student determines if the converse of a conditional statement is true or false.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.C. The student demonstrates what it means to prove mathematically that statements are true.
b.3.E. The student uses deductive reasoning to prove a statement.

Background: Participants should be familiar with definitions and properties of quadrilaterals, conditional statements, and biconditional statements.

Materials: easel paper, colored markers

## New Terms:

## Procedures:

We have an exhaustive list of the properties for each quadrilateral. This exhaustive list defines the quadrilateral. In this activity we will identify the properties that are sufficient
to alternately define the particular quadrilateral. There are many ways of combining properties to sufficiently define a particular quadrilateral.

Work through the following example with the group.
Consider the rhombus:

1. State the basic definition of the rhombus as a conditional statement. If a quadrilateral is a rhombus, then it has four congruent sides.
2. List properties of the rhombus as conditional statements.

- If a quadrilateral is a rhombus, then its opposite sides are parallel.
- If a quadrilateral is a rhombus, then its opposite sides are congruent.
- If a quadrilateral is a rhombus, then its opposite angles are congruent.
- If a quadrilateral is a rhombus, then its consecutive angles are supplementary.
- If a quadrilateral is a rhombus, then its diagonals bisect the vertex angles.
- If a quadrilateral is a rhombus, then its diagonals are perpendicular to each other.
- If a quadrilateral is a rhombus, then its diagonals bisect each other.

3. Select no more than two properties which can be combined to sufficiently define the rhombus alternately.
In this example we will select perpendicular diagonals and bisecting diagonals.
4. Write a proof to show that the properties selected in 3 sufficiently define the rhombus.

Diagram:


Given: Quadrilateral $A B C D$ with diagonals $\overline{A C}$ and $\overline{B D}$ intersecting at point $E$, $\overline{A C} \perp \overline{B D}, \overline{E A} \cong \overline{E C}$ and $\overline{E B} \cong \overline{E D}$
Prove: Quadrilateral $A B C D$ is a rhombus.

$$
\overline{A C} \perp \overline{B D}
$$

$$
\angle A E B \cong \angle B E C \cong \angle C E D \cong \angle D E A
$$

$$
\overline{E A} \cong \overline{E C} \text { and } \overline{E B} \cong \overline{E D}
$$

(Given)
(Perpendicular lines intersect to form four congruent right angles.)
(Given)

$$
\begin{equation*}
\triangle A E B \cong \triangle A E D \cong \triangle C E D \cong \triangle C E B \tag{SAS}
\end{equation*}
$$

Quadrilateral $A B C D$ is a rhombus

$$
\overline{A B} \cong \overline{D A} \cong \overline{C D} \cong \overline{B C}
$$

(Corresponding sides of congruent triangles are congruent.)
(Definition of a rhombus)
5. If the given properties selected do provide an alternate definition, then rewrite the alternate definition as a biconditional statement, using "if and only if" which implies that both the conditional and its converse are true.
A quadrilateral is a rhombus if and only if the diagonals are perpendicular to each other and bisect each other.

Participants now work in groups, each group focusing on a different quadrilateral from the activity sheet. Provide each group with easel paper and markers. Participants develop an alternate definition for their quadrilateral following the method used above. Each group presents its work on easel paper to the entire group.

The definitions and properties are listed below. Since there are a variety of ways to combine properties to find alternate definitions, answers may vary. The group which works on the rhombus should combine different pairs of properties from those used in the example.

Square

1. State the basic definition of the square as a conditional statement.

If a quadrilateral is a square, then it has four congruent sides and four right angles.
2. List properties of the square as conditional statements.

- If a quadrilateral is a square, then its opposite sides are parallel.
- If a quadrilateral is a square, then its sides are congruent.
- If a quadrilateral is a square, then its consecutive sides are perpendicular.
- If a quadrilateral is a square, then its diagonals are congruent to one another.
- If a quadrilateral is a square, then its diagonal bisect each other at right angles.
- If a quadrilateral is a square, then its diagonals bisect the vertex angles.
- If a quadrilateral is a square, then its angles are congruent right angles.
- If a quadrilateral is a square, then its consecutive angles are congruent and supplementary.
- If a quadrilateral is a square, then its opposite angles are congruent and supplementary.

Kite

1. State the basic definition of the kite as a conditional statement.

If a quadrilateral is a kite, then it has two distinct pairs of consecutive congruent sides.
2. List properties of the kite as conditional statements.

- If a quadrilateral is a kite, then its opposite sides are not congruent.
- If a quadrilateral is a kite, then its diagonals are perpendicular to each other.
- If a quadrilateral is a kite, then it has one pair of congruent opposite angles.
- If a quadrilateral is a kite, then only one of its diagonals is bisected by the other diagonal, and the bisected diagonal has its endpoints on the congruent angles' vertex.

Parallelogram

1. State the basic definition of the parallelogram as a conditional statement. If a quadrilateral is a parallelogram, then it has two pairs of parallel sides.
2. List properties of the parallelogram as conditional statements.

- If a quadrilateral is a parallelogram, then its opposite sides are congruent.
- If a quadrilateral is a parallelogram, then its opposite angles are congruent.
- If a quadrilateral is a parallelogram, then its consecutive angles are supplementary.
- If a quadrilateral is a parallelogram, then its diagonals bisect one another.


## Rectangle

1. State the basic definition of the rectangle as a conditional statement.

If a quadrilateral is a rectangle, then it has four right angles.
2. List properties of the rectangle as conditional statements.

- If a quadrilateral is a rectangle, then its opposite sides are congruent.
- If a quadrilateral is a rectangle, then its opposite sides are parallel.
- If a quadrilateral is a rectangle, then its opposite angles are congruent.
- If a quadrilateral is a rectangle, then its opposite angles are supplementary.
- If a quadrilateral is a rectangle, then its consecutive angles are congruent.
- If a quadrilateral is a rectangle, then its consecutive angles are supplementary.
- If a quadrilateral is a rectangle, then its diagonals are congruent.
- If a quadrilateral is a rectangle, then its diagonals bisect one another.

Ask groups to present their work to close the activity.
Success with this activity indicates that participants are working at the Deductive Level, because they created deductive proofs.

## Alternate Definitions

Pick one of the following quadrilaterals: rhombus, square, kite, parallelogram, or rectangle.

1. State the basic definition of the quadrilateral as a conditional statement.
2. List properties of the quadrilateral as conditional statements.
3. Select no more than two properties which can be combined to sufficiently define the quadrilateral alternately.
4. Write a proof to show that the properties selected in 3 sufficiently define the quadrilateral.

## Diagram:

Given:

Prove:

Proof:
5. If the given properties selected do provide an alternate definition, then rewrite the alternate definition as a biconditional statement, using "if and only if" which implies that both the conditional and its converse are true.

## Circle Proofs

Overview: Participants prove theorems about inscribed angles.

> TExEective: Mathematics Competencies
> III.012.A. The beginning teacher understands axiomatic systems and their components (e.g., undefined terms, defined terms, theorems, examples, counterexamples).
> III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
> III.013.B The beginning teacher analyzes the properties of circles and the lines that intersect them.
> III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
> V.018.A. The beginning teacher understand the nature of proof, including indirect proof, in mathematics.
> V.018.B. The beginning teacher applies correct mathematical reasoning to derive valid conclusions from a set of premises.
> V.018.C. The beginning teacher uses inductive reasoning to make conjectures and uses deductive methods to evaluate the validity of conjectures.
> V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

## Geometry TEKS

b.1.A. The student develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning and theorems.
b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.C. The student demonstrates what it means to prove mathematically that statements are true.
b.3.E. The student uses deductive reasoning to prove a statement.
e.2.C. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of circles and the lines that intersect them.

## Background: Participants need to be familiar with exterior-interior angle

 relationships in triangles.
## Materials:

New Terms: intercepted arc

## Procedures:

We formally introduce circle properties in a later unit, but we will explore them in this unit with theorems taken from Discovering Geometry: An Investigative Approach, $3^{\text {rd }}$ Edition, © 2003, pp. 325-327 with permission from Key Curriculum Press.

Participants should recall the definitions of central angle and inscribed angle. A central angle has its vertex at the center of the circle. An inscribed angle has its vertex on the circle and its sides are chords.

We define intercepted arc of a circle as the part of a circle whose endpoints are the points where the segments of a central angle intersect the circle.

In the following proofs, we use the fact that the measure of a central angle is equal to the measure of its intercepted arc. In the diagram below $\mathrm{m} \angle B A C=\mathrm{m} B C$, i.e., if $\mathrm{m} \angle B A C$ $=25^{\circ}$, then $\mathrm{m} B C=25^{\circ}$ and visa versa.


Remind participants to add the term intercepted arc to their glossaries.

1. Show that the measure of an inscribed angle $(\angle M D R)$ in a circle equals half the measure of its central angle $(\angle M O R)$ that intercepts the same $\operatorname{arc}(R M)$ when a side of the angle, $\overline{D R}$, passes through the center of the circle.


R

Given: Circle $O$ with inscribed $\angle M D R$ on diameter $\overline{D R}$
Prove: $m \angle M D R=\frac{1}{2} m \angle M O R$
$m \angle M O R=m M R$
$m \angle M O R=m \angle M D O+m \angle D M O$
$\overline{D O} \cong \overline{M O}$
$\Delta D M O$ is isosceles
$m \angle M D O=m \angle D M O$
$m \angle M O R=m \angle M D O+m \angle M D O$
$m \angle M O R=2 m \angle M D O$
$m \angle M D O=\frac{1}{2} m \angle M O R$
(The measure of a central angle is equal to the measure of its intercepted arc.)
(The measure of the remote exterior angle of a triangle is equal to the sum of the measures of the two opposite interior angles of the triangle.)
(Radii of the same circle are congruent.)
(Definition of isosceles triangle.)
(Base angles of an isosceles triangle are congruent.)
(Substitution)
(Combine like terms)
(Divide by 2)
This proves that the measure of the inscribed angle in a circle equals half the measure of its central angle that intercepts the same arc.
2. Show that the measure of an inscribed angle in a circle equals half the measure of its intercepted arc when the center of the circle is outside the angle.


Given: Circle $O$ with inscribed $\angle M D K$ on one side of diameter $\overline{D R}$
Prove: $m \angle M D K=\frac{1}{2} K M$
$m \angle K D R=m \angle M D R+m \angle M D K \quad$ (Angle Addition)
$m \angle M D K=m \angle K D R-m \angle M D R$
(Subtract $m \angle M D R$ )
$m \angle M D R=\frac{1}{2} M R$ and $m \angle K D R=\frac{1}{2} K R \quad$ (We proved that the measure of an inscribed angle in a circle equals half the measure of its central angle when a side of the angle passes through the center of the circle.)
$m K R=m M R+m K M$
(Arc Addition)
$m \angle M D K=\frac{1}{2}(M R+K M)-\frac{1}{2} M R$
(Substitution)
$m \angle M D K=\frac{1}{2} K M$
(Simplify)
Therefore the measure of an inscribed angle in a circle equals half the measure of its intercepted arc when the center of the circle is outside the angle.
3. Show that the measure of an inscribed angle in a circle equals half the measure of its intercepted arc when the center of the circle is inside the angle.


Given: Circle $O$ with inscribed angle $\angle M D K$
Prove: $m \angle M D K=\frac{1}{2} m M R K$
$m \angle M D K=m \angle M D R+m \angle R D K$
(Angle Addition)

$$
\left.\begin{array}{l}
m \angle M D R=\frac{1}{2} m M R \text { and } m \angle R D K=\frac{1}{2} m R K \quad \begin{array}{l}
\text { (The measure of an inscribed angle } \\
\text { in a circle equals half the measure of } \\
\text { its central angle when a side of the } \\
\text { angle passes through the center of } \\
\text { the circle.) }
\end{array} \\
m \angle M D K=\frac{1}{2} m M R+\frac{1}{2} m R K \\
\text { (Substitution) }
\end{array}\right\} \begin{aligned}
& \text { (Simplify) } \\
& m \angle M D K=\frac{1}{2}(m M R+m R K) \\
& m M R+m R K=m M R K
\end{aligned} \begin{aligned}
& \text { (Arc Addition) }
\end{aligned}
$$

4. Show that inscribed angles that intercept the same arc are congruent.


Given: Circle $O$ with $\angle C A B$ and $\angle B D C$ inscribed in $B C$
Prove: $m \angle C A B=m \angle B D C$
$m \angle C A B=\frac{1}{2} B C$ and $m \angle B D C=\frac{1}{2} B C \quad$ (The measure of an inscribed angle in a circle equals half the measure of its intercepted arc.)
$m \angle C A B=m \angle B D C$
(Substitution)
Therefore inscribed angles that intercept the same arc are congruent.
5. Show that the angles inscribed in a semicircle are right angles.


Given: $\angle A C B$ is an inscribed angle, $\overline{A D B}$ is a diameter Prove: $m \angle A C D=90^{\circ}$
$\angle A C B$ is an inscribed angle
$\overline{A D B}$ is a diameter
$m \angle A D B=180^{\circ}$
$m \angle A C D=\frac{1}{2} m \angle A D B$
$m \angle A C D=90^{\circ}$
(Given)
(Given)
(A straight angle measures $180^{\circ}$.)
(The measure of an inscribed angle
in a circle equals half the measure of its intercepted arc.)
(Substitution)

Therefore, the measure of an inscribed angle in a semicircle is $\frac{1}{2}\left(180^{\circ}\right)$ or $90^{\circ}$.
Success in this activity indicates that participants are at the Deductive Level, because they formally develop deductive proofs.

## Circle Proofs

Write the given statements and those that are to be proved. Then write the proof itself.

1. Show that the measure of an inscribed angle ( $\angle M D R$ ) in a circle equals half the measure of its central angle ( $\angle M O R$ ) that intercepts the same arc ( $R M$ ) when a side of the angle, $\overline{D R}$, passes through the center of the circle.

2. Show that the measure of an inscribed angle in a circle equals half the measure of its intercepted arc when the center of the circle is outside the angle.

3. Show that the measure of an inscribed angle in a circle equals half the measure of its intercepted arc when the center of the circle is inside the angle.

4. Show that inscribed angles that intercept the same arc are congruent.

5. Show that the angles inscribed in a semicircle are right angles.


## References and Additional Resources

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## Unit 5 - Area

## What Is Area?

Overview: Participants determine the area of a rectangle by counting the number of square units needed to cover the region. Group discussion deepens participants' understanding of area (number of square units needed to cover a given region) and connects the formula for the area of a rectangle to the underlying array structure.

## Objective: TExES Mathematics Competencies

III.013.D. The beginning teacher computes the perimeter, area, and volume of figures and shapes created by subdividing and combining other figures and shapes (e.g., arc length, area of sectors).
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

## Geometry TEKS

b.3.D. The student uses inductive reasoning to formulate a conjecture. e.1.A. The student finds area of regular polygons and composite figures.

Background: No prerequisite knowledge is necessary for this activity.
Materials: index cards, patty paper, straightedge
New Terms: area

## Procedures:

Background information: In order to accurately count the units of a given space, it is necessary to mentally organize the space in a structured manner. However, research referenced in Schifter, Bastable, and Russell (2002), on how children learn the concept of area supports the theory that the structure of the rectangular array is not intuitively obvious to children. When asked to cover a rectangular region, children progress from incomplete or unsystematic coverings to individually drawn units to the use of a row or column iteration. Gradually they will rely less on drawing and move towards multiplication or repeated addition. Covering a rectangular region with unit squares helps children understand area measure, but they must ultimately be able to formally connect area, linear measurement, and multiplication in order to truly understand the area formula $A=b \cdot h$. According to Schifter, Bastable, and Russell (2002), the drawing, filling, and counting that children use in this developmental process are both motor and mental actions that coordinate to organize spatial structuring.

What implications does this research have for secondary teachers? Many of our students come to us with an understanding of area at the Visual Level of the van Hiele model of
geometric development. They recognize figures by their shape and understand area as the blank space within the boundaries. Others have moved to the Descriptive Level, which implies that they can mentally see the rectangular array which overlays the shape. This activity asks participants to draw the grid, rather than giving them a pre-structured grid in order for them to experience the type of activity necessary to move a student from the Visual to the Descriptive Level. Secondary students functioning on the Relational Level are able to compare linear dimensions with grid areas and apply formulas with understanding. Teachers must be aware that although students may become proficient at rote application of formulas, they may not be functioning at the Relational Level. If they have not been given sufficient opportunity to understand the principles behind the formulas, they will have difficulty modifying a procedure to fit a particular situation as is necessary to find areas of composite figures and shaded regions.

Distribute index cards to participants and ask them to write a response to the question "What is area?". Indicate that they will have an opportunity to share and revise their responses at the conclusion of this activity. Then, allow time for participants, working independently or in pairs, to complete the activity using the patty paper or straight edge to determine the number of square units needed to completely cover the rectangular region.

Note that the given square units are not convenient measures, such as $1 \mathrm{~cm}^{2}$ or $1 \mathrm{in}^{2}$. Consequently, participants will be less likely to simply measure the rectangle and use the area formula without having the experience of drawing the units. Whether participants mark off the units on two adjacent sides of the rectangle and multiply or actually draw in one or more rows and columns of units, they will be counting the units by considering how many rows of squares are needed to cover the region.

1. Determine the number of square units needed to cover this rectangular region.


The rectangle measures $6 \cdot 8$ square units. Therefore, it will take 48 square units to cover the rectangle.
2. Determine the number of square units needed to cover this rectangular region. (Same rectangle, different square unit)

1 square unit


The rectangle measures $9 \cdot 12$ square units. Therefore, it will take 108 square units to cover the rectangle.

Was it necessary to draw all 108 square units to determine that it would take 108 units to cover the rectangular region?
No. After drawing one row and one column of square units the total number of squares can be obtained by considering how many rows and columns of squares will be needed to cover the entire region.

## Has the activity caused you to reconsider your definition of area?

Some participants may have responded to the question "What is area?" by stating that area is the amount of space covered by a particular region. It is important to make the distinction that area is the number of square units needed to completely cover a particular region. If the same figure is measured in different units, the number representing the area of the region will be different, but the area will remain constant.

A more abstract definition of area, provided by Michael Serra (Serra, 2003) states that area is a function that assigns to each two-dimensional geometric shape a nonnegative real number so that (1) the area of every point is zero, (2) the areas of congruent figures are equal, and (3) if a shape is partitioned into sub regions, then the sum of the areas of those sub regions equals the area of the shape.

## If a figure is rotated so that a different side is considered the base, will the area formula necessarily give the same result?

Yes. Surprisingly, the answer to this question is not evident to all students. According to the work of Clements and Battista referenced in Schifter, Bastable, and Russell (2002), orientation, the position of objects in space in relation to an external frame of reference, is for some children a part of their definition of a particular shape. If secondary students have not had adequate experience manipulating by rotating, flipping or sliding shapes, they may be working at the Visual Level with an inadequate understanding of shape. In developing an understanding of area, students should observe that rotations, reflections and translations preserve area while dilations do not.

## What Is Area?

1. Determine the number of square units needed to cover this rectangular region.


1 square unit
2. Determine the number of square units needed to cover this rectangular region. (Same rectangle, different square unit)
$\square$
1 square unit


## Investigating Area Formulas

Overview: | Participants cut and rearrange two triangles, a parallelogram, and a kite |
| :--- |
| to form rectangles with the same areas. Examination of the points at |
| which figures must be cut will lead to a deeper understanding of the |
| formula for the area of each figure. |

## Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
b.3.C. The student demonstrates what it means to prove mathematically that statements are true.
d.2.C. The student develops and uses formulas including distance and midpoint.

Background: Participants should know the formula for the area of a rectangle and be able to identify the base and altitude of a triangle, the base and altitude of a parallelogram, and the diagonals of a kite.

Materials: transparency sheets, colored pencils, glue or tape, patty paper, scissors

## New Terms:

## Procedures:

Participants sit in groups of 3-4 to work collaboratively. However, each participant should cut and glue his/her own figures.

Can any parallelogram, triangle or kite be cut and rearranged to form a rectangle of the same area?
Allow time for discussion within groups and then ask two or three groups to share their responses with the entire group. It is fairly obvious that any parallelogram can be cut to
form a rectangle of equal area, but the group may or may not be able to reach a consensus on the triangle and kite.

Briefly describe the activity. Each of the figures on the activity sheet has the same area as the rectangle. Using a colored pencil, trace the parallelogram on patty paper. Then, using the least number of cuts possible, cut the parallelogram and rearrange the pieces to form a rectangle of equal area. The rectangle will help you determine where to cut. Lay the patty paper tracing over the rectangle and slide it around to decide where to cut. Using a different colored pencil, draw the cut line on the patty paper figure and then cut. Assemble the pieces to form a rectangle and glue it next to the original parallelogram. Repeat the process for each of the figures.

While the groups are working, assign each of the figures to a different participant to draw on a transparency for use during the group discussion. When most participants have completed the task, reconvene as a large group for discussion.

Parallelogram:


Can we express the dimensions of the rectangle in terms of the dimensions of the parallelogram?
Yes. The base of the rectangle is the base of the parallelogram. The height of the rectangle is the height of the parallelogram.

What does this tell us about the formula for the area of the parallelogram? Since the area of the parallelogram is equal to the area of the rectangle, the area of the parallelogram is $b \cdot h$.

Obtuse Triangle:


How would you describe the location of the cut lines on the obtuse triangle?
The cuts must pass through the midpoints of the sides of the triangle as shown.
What do we call the segment that connects the midpoints of the sides of a triangle? The midsegment

What do we know about the midsegment of a triangle?
The midsegment is parallel to the base and one half the length of the base of the triangle.
Can we express the dimensions of the rectangle in terms of the dimensions of the triangle?
Yes. The length of the base of the rectangle is equal to the length of the midsegment of the triangle, $m$. The height of the rectangle, $h$, is the height of the triangle.

Can we use this information to derive the formula for the area of a triangle?
Since we know
Area of triangle $=$ Area of rectangle

$$
=b \cdot h
$$

By substitution,
Area of triangle $=m \cdot h$
By the definition of a midsegment,

$$
\begin{aligned}
\text { Area of triangle } & =\frac{1}{2}(2 b) \cdot h \\
& =\frac{1}{2}(\text { base of triangle })(\text { height })
\end{aligned}
$$

Acute Triangle:


How would you describe the location of the cut lines on the acute triangle?
One cut line goes through the midsegment of the triangle and one cut line is the altitude joining the midsegment to the opposite vertex of the triangle.

Can we use this information to derive the formula for the area of a triangle?
Since we know
Area of triangle $=$ Area of rectangle

$$
\begin{aligned}
& =b \cdot h \\
& =b \cdot \frac{1}{2}(2 h) \\
& =\frac{1}{2}(\text { base of triangle }) \text { (height of triangle) }
\end{aligned}
$$

Kite:


How can we use the formula we have derived for the area of a triangle to derive the formula for the area of the kite?
$d_{1}$ lies on the line of symmetry for the kite. The two triangles formed by the line of symmetry, $d_{l}$, are congruent.
$\frac{1}{2}\left(d_{2}\right)$ is the length of the altitude of each triangle, and $\frac{1}{2}\left(d_{2}\right)=h$
Area of one triangle $=\frac{1}{2}\left(d_{1}\right) \cdot h=\left(\frac{1}{2}\right)\left(d_{1}\right)\left(\frac{1}{2}\right)\left(d_{2}\right)$
Area of kite $=2\left(\frac{1}{2}\right)\left(d_{1}\right)\left(\frac{1}{2}\right)\left(d_{2}\right)=\frac{1}{2}\left(d_{1} \cdot d_{2}\right)$

$$
=\frac{1}{2} d_{1} \cdot d_{2}
$$

Success in this activity indicates that participants are working at the Relational Level because they must discover the relationship between the area rule for a rectangle and the area rule for a parallelogram, triangle, or kite. While a participant at the Descriptive Level will be able to cut the figures to form the rectangle of the same area, he/she will need prompting to explain "why it works" using informal deductive arguments.

## Investigating Area Formulas

Trace the parallelogram, triangles, and kite on patty paper. Then cut and arrange the pieces of each figure to form a rectangle congruent to the given rectangle. Glue each new figure next to the original figure from which it was made. Label the dimensions of the rectangle $b$ and $h$. Determine the area for each figure in terms of $b$ and $h$.



## Area of Trapezoids

Overview: Participants find many different ways to derive the formula for the area of a trapezoid.

Objective: TExES Mathematics Competencies<br>III.011.A. The beginning teacher applies dimensional analysis to derive units and formulas in a variety of situations (e.g., rates of change of one variable with respect to another and to find and evaluate solutions to problems.<br>III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).<br>V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

## Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
b.3.C. The student demonstrates what it means to prove mathematically that statements are true.
d.2.C. The student develops and uses formulas including distance and midpoint.

Background: Participants need to know the formulas for areas of triangles, rectangles, and parallelograms. A clear understanding of the midsegment of a trapezoid is a prerequisite.

Materials: patty paper, scissors

## New Terms:

## Procedures:

Distribute the activity sheet and allow about 10 minutes for participants to work on it individually so that each will have an opportunity for independent discovery. Within their group of four, allow another 15 minutes for participants to share all of the methods for deriving the formula for the area of a trapezoid that they have discovered. Then jigsaw the groups and allow about 20 minutes for the new groups to share their work. During the final 15 minutes of the activity, participants return to their original groups and share any new method they have seen during the jigsaw.

Scaffolding questions to help participants who are having difficulty finding multiple solutions:

## Can you use a technique from the previous activity on areas of triangles and parallelograms?

In the previous activity we saw that the midsegment could be used to cut and rearrange a triangle into a rectangle of the same area.

Can a trapezoid be divided into figures whose area formulas are known?
Yes, it can be divided into a rectangle and two triangles, three triangles, or four triangles.
Can you create a larger figure in which the trapezoid is one of the composite parts? Yes, the trapezoid can be copied and rotated $180^{\circ}$ to create a parallelogram whose area is equal to twice the area of the original trapezoid. A triangle can be drawn such that a rectangle is formed from the triangle and the original trapezoid.

Answers will vary. Seven derivations are given below.

1. Base of rectangle $=$ Midsegment of trapezoid

Height of rectangle $=$ Height of trapezoid Area of trapezoid $=$ Area of rectangle $=$ Midsegment of trapezoid • height of trapezoid

2. Area of triangle $1=\frac{1}{2}$ ah

Area of triangle $2=\frac{1}{2}$ ch
Area of rectangle $=b h$

$$
\begin{aligned}
\text { Area of trapezoid } & =\frac{1}{2} a h+\frac{1}{2} c h+b h \\
& =h\left(\frac{1}{2} a+\frac{1}{2} c+b\right) \\
& =\frac{1}{2} h(a+c+2 b) \\
& =\frac{1}{2} h[(a+c+b)+b] \\
& =\frac{1}{2} \text { (height) (sum of bases) }
\end{aligned}
$$


3. Area of trapezoid $=$ Area of parallelogram + area of triangle

$$
\begin{aligned}
& =h \cdot b_{2}+\frac{1}{2} h\left(b_{1}-b_{2}\right) \\
& =h \cdot b_{2}+\frac{1}{2} h \cdot b_{1}-\frac{1}{2} h \cdot b_{2} \\
& =\left(\frac{1}{2}\right) h\left(2 b_{2}+b_{1}-b_{2}\right) \\
& =\left(\frac{1}{2}\right) h\left(b_{1}+b_{2}\right) \\
& =\left(\frac{1}{2}\right)(\text { height })(\text { sum of bases })
\end{aligned}
$$


4. Area of trapezoid $=\left(\frac{1}{2}\right) h a+\left(\frac{1}{2}\right) h \cdot b_{1}+\left(\frac{1}{2}\right) h c$

$$
\begin{aligned}
& =\left(\frac{1}{2}\right) h\left(a+b_{1}+c\right) \\
& =\left(\frac{1}{2}\right) h\left(b_{1}+b_{2}\right) \\
& \left.=\left(\frac{1}{2}\right) \text { (height }\right) \text { (sum of bases) }
\end{aligned}
$$


5. Area of trapezoid $=\left(\frac{1}{2}\right) h_{1} \cdot b_{1}+\left(\frac{1}{2}\right) h_{1} \cdot b_{2}+\left(\frac{1}{2}\right) h_{1} \cdot m+\left(\frac{1}{2}\right) h_{1} \cdot m$

$$
\begin{aligned}
& =\left(\frac{1}{2}\right) h_{1}\left(b_{1}+b_{2}\right)+h_{1} \cdot m \\
& =h_{1} \cdot m+h_{1} \cdot m \\
& =2 h m
\end{aligned}
$$


6. Area of trapezoid $=$ Area of parallelogram $=2 \cdot m \cdot h_{1}=2 m \cdot\left(\frac{1}{2}\right) h=m h$

7. Area of trapezoid $=\left(\frac{1}{2}\right)$ Area of parallelogram
$=\left(\frac{1}{2}\right) h\left(b_{1}+b_{2}\right)$
$=($ midsegment of trapezoid $)$ (height of trapezoid $)$
$b_{2}$
$b_{1}$


Success in this activity indicates that participants are working at the Relational Level because they must show the relationships among area rules and give informal deductive arguments to justify the rule determining the area of a trapezoid.

## Area of Trapezoids

How many ways can you derive the formula for the area of a trapezoid?


## Area of Circles

Overview: Participants develop the concept of area of circles by drawing a circle and then cutting it into sectors. The sectors are rearranged to form a shape that resembles a parallelogram.

## Objective: TExES Mathematics Competencies

III.011.B. The beginning teacher applies formulas for perimeter, area, surface area, and volume of geometric figures and shapes (e.g., polygons, pyramids, prisms, cylinders, cones, spheres) to solve problems.
III.013.D. The beginning teacher computes the perimeter, area, and volume of figures and shapes created by subdividing and combining other figures and shapes (e.g., arc, length, area of sectors). V.018.E. The beginning teacher understands the problem-solving process (i.e., recognizing that a mathematical problem can be solved in a variety of ways, selecting an appropriate strategy, evaluating the reasonableness of a solution).
V.019.C. The beginning teacher translates mathematical ideas between verbal and symbolic forms.
V.019.D. The beginning teacher communicates mathematical ideas using a variety of representations (e.g., numeric, verbal, graphical, pictorial, symbolic, concrete).

## Geometry TEKS

b.3.C. The student demonstrates what it means to prove mathematically that statements are true.
b.4. The student selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems. d.2.C. The student develops and uses formulas including distance and midpoint.
e.1.B. The student finds areas of sectors and arc lengths of circles using proportional reasoning.

Background: Participants need to understand area as a covering of a space, pi, and the circumference of a circle.

Materials: cups (preferably large plastic cups), glue or tape, graphing calculator, colored markers, patty paper, scissors

New Terms: circumference, sector

## Procedures:

Participants work in pairs for both parts of this activity.

## Part 1: Developing the Formula for the Area of a Circle

Using a piece of patty paper, trace the circular rim of the plastic cup. Fold the patty paper to locate the center of the circle. Fold the circle in half four additional times to create sixteenths. Firmly crease the folds. Using a marker, color half of the circle in one color and the other half of the circle in a different color.


Cut out the 16 sectors of the circle. Paste the sectors in one color upside down on a sheet of paper.


Paste the sectors in the other color between the sectors in an interlocking manner, forming a parallelogram shaped figure.


Possible scaffolding questions to help participants develop the formula of area:
How is the sum of the sectors related to the area of the circle?
The sum of the areas of the sectors is equal to the area of the circle.
Identify the length of the parallelogram.
The length is half of the circumference of the circle. Because the circumference is $\mathrm{C}=2 \pi r$, and the length of the parallelogram is half of the circle, the length is $\pi r$. Label the length $\pi r$.

## Identify the width of the parallelogram.

The width is the same as the radius of the circle. Label it $r$.

## What is the area of the parallelogram-shaped figure?

The area is $(\pi r)(r)=\pi r^{2}$.

## Is the area of a circle a function of the radius?

Yes; the area depends on the length of the radius.

## Part 2: Applying the Formula for the Area of a Circle

1. The circumference of a circle and the perimeter of a square are each 20 in . Which has the greater area, the circle or the square? How much greater?

$$
C=2 \pi r
$$


$20=2 \pi r$
$r \approx \frac{10}{\pi}$

$20=4 s$
$A=\pi r^{2}$
$5=s$
$A=\pi \times\left[\frac{10}{\pi}\right]^{2}$
$A \approx 31.8 \mathrm{in}^{2}$

The circle has the greater area. The approximate difference is $31.8 i^{2}-25 i i^{2}=6.8 i n^{2}$.
2. A rice farmer uses a rotating irrigation system in which 8 sections of 165 -foot pipe are connected end to end. The resulting length of sprinkler pipe travels in a circle around a well, which is located in the center of the circle.

a. How many acres of the square field are irrigated? (one acre is $43,560 \mathrm{ft}^{2}$ ) There are 8 sections of 165 feet pipe, or 1320 feet of pipe, which is the radius of the irrigated field. The area is $\pi \cdot 1320^{2}$ or $1742400 \pi \mathrm{ft}^{2}$ or $5473911.0 \mathrm{ft}^{2}$. To find the number of acres, divide the total square feet by the $43,560 \mathrm{ft}^{2}$ in one acre.
$\frac{5473911}{43560} \approx 125.7$ acres .
b. Broccoli is often planted in the corners of the field, which are not irrigated. How many acres of broccoli are planted in this field?

Area of square - Area of circle $=$ Area of corners

The diameter of the circle is the length of the side of the square. The diameter is 2640 $f t$. The area of the square is $2640^{2}$ or 6969600 is ft ${ }^{2}$.

$$
\text { Area of square }- \text { Area of circle }=\text { Area of corners }
$$

Using Substitution,

$$
6969600-5473911=1495689 f t^{2}
$$

Changing the area to acres,

$$
\frac{1495689}{43560} \approx 34.3 \text { acres }
$$

3. Ron opened a new restaurant. He asked his wait staff to determine the greatest number of cylindrical glasses that would fit on a 24 in . by 12 in . tray. Each glass has a radius of 1.5 inches. Being a math geek, he began to wonder how much area was not being used on the tray. Make a drawing of the arrangement that fits the most glasses on the tray. Make a prediction of how many glasses will fit on the tray. What is the area on the tray not used by the cylindrical glasses?


Area of glass $=\pi(1.5)^{2} \approx 7.1 \mathrm{in}^{2}$
Area of 32 glasses $=32(7.1) \approx 227.2 \mathrm{in}^{2}$
Area of tray $=24(12)=288 \mathrm{in}^{2}$
Area of difference $=288 \mathrm{in}^{2}-227.2 \mathrm{in}^{2}=60.8 \mathrm{in}^{2}$
4. When Ruth's family moved during her sophomore year of high school, they stayed in a motel for 3 weeks. Their dog, Frosty, was tied at the corner of the motel on a 50 foot rope. If the motel were a 40 ft by 30 ft rectangle, how many square feet could Frosty wander? (Give answer in terms of $\pi$.)
The dog could run in a three quarters circle having a radius of 50 feet.
At both ends of the building his path forms a quarter circle, one with a radius of 20 feet, and the
 other with a radius of 10 feet. The dog can wander $2000 \pi$ square feet.
5. Circle Pizzeria is changing the size of its circular pizza from 12 inches to 16 inches and increasing the number of slices per pizza from 8 to 10 . What is the percent increase of the size of each new slice? (Round answer to the nearest tenth of a percent.)


12 inch pizza
Area of 12 inch pizza $=\pi 6^{2}=36 \pi$
$\frac{36 \pi}{8 \text { slices }}=4.5 \pi$ in. $^{2}$ per slice


16 inch pizza
Area of 16 inch pizza $=\pi 8^{2}=64 \pi$
$\frac{64 \pi}{10 \text { slices }}=6.4 \pi$ in. ${ }^{2}$ per slice

$$
6.4 \pi-4.5 \pi=1.9 \pi \text { difference }
$$

$$
\frac{1.9 \pi}{4.5 \pi} \approx 42.2 \% \text { increase }
$$

6. Circles of radius 4 with centers at $(4,0)$ and $(0,4)$ overlap in the shaded region shown in the figure. Find the area of the shaded region in terms of $\pi$. One way to find the area of the shaded region is to draw the diagonal of the square. $\frac{1}{2}$ Area of shaded region $=$ Area of quarter circle - Area of triangle

$$
\text { Area of sector: } \begin{aligned}
& \frac{1}{4} \pi r^{2}-\frac{1}{2} b h \\
& \frac{1}{4} \pi 16-\frac{1}{2} 16 \\
& 2(4 \pi-8) \\
& 8 \pi-16 \text { units }^{2}
\end{aligned}
$$


7. In the figure shown, all arcs are semicircles, and those that appear to be congruent are. What is the area of the shaded region? (Give answer in terms of $\pi$.)


$$
\begin{aligned}
& 2(\text { Area midsized semi-circle })=2\left(\frac{1}{2} \pi \cdot 2^{2}\right)=4 \pi \\
& 4(\text { Area small semi-circle })=4\left(\frac{1}{2} \pi \cdot 1^{2}\right)=2 \pi
\end{aligned}
$$

Area midsized semi-circles. Area small semi-circles $=4 \pi-2 \pi=2 \pi$
Remind participants to add the new terms circumference and sector to the glossaries.

Part 1: Success in this part of the activity indicates that participants are working at the Relational Level because they must discover the relationship between the formula for the area of a parallelogram and the formula for the area of a circle.

Part 2: Success in this part of the activity indicates that participants are working at the Relational Level because they must solve geometric problems by using known properties of figures and insightful approaches.

## Area of Circles

Solve the following problems.

1. The circumference of a circle and the perimeter of a square are each 20 in . Which has the greater area, the circle or the square? How much greater?
2. A rice farmer uses a rotating irrigation system in which 8 sections of 165 foot pipe are connected end to end. The resulting length of sprinkler pipe travels in a circle around a well, which is located in the center of the circle.
a. How many acres of the square field are irrigated? (One acre is $43,560 \mathrm{ft}^{2}$.)

b. Broccoli is often planted in the corners of the field, which are not irrigated. How many acres of broccoli are planted in this field?
3. Ron opened a new restaurant. He asked the wait staff how many cylindrical glasses would fit on a 24 in . by 12 in. tray. Each glass has a radius of 1.5 inches. Being a math geek, he began to wonder how much area was not being used on the tray. Make a drawing of the arrangement that fits the most glasses on the tray. Make a prediction of how many glasses will fit on the tray. What is the area on the tray not used by the cylindrical glasses?
4. When Ruth's family moved during her sophomore year of high school, they stayed in a motel for 3 weeks. Their dog, Frosty, was tied at the corner of the motel on a 50 foot rope. If the motel was a 40 ft by 30 ft rectangle, how many square feet could Frosty wander? (Give answer in terms of $\pi$.)

5. Circle Pizzeria is changing the size of its circular pizza from 12 inches to 16 inches and increasing the number of slices per pizza from 8 to 10 . What is the percent of increase of the size of each new slice? (Round answer to nearest tenth of a percent.)
6. Circles of radius 4 with centers at $(4,0)$ and $(0,4)$ overlap in the shaded region shown in the figure. Find the area of the shaded region in terms of $\pi$.

7. In the figure shown, all arcs are semicircles, and those that appear to be congruent are. What is the area of the shaded region? (Give answer in terms of $\pi$.)


## Applying Area Formulas

Overview: Participants use the problem solving process to find the area of composite figures (composite of triangles, quadrilaterals and circles).

## Objective: TExES Mathematics Competencies <br> III.011.B. The beginning teacher applies formulas for perimeter, area, surface area, and volume of geometric figures and shapes (e.g., polygons, pyramids, prisms, cylinders, cones, spheres) to solve problems). <br> III.013.D. The beginning teacher computes the perimeter, area, and volume of figures and shapes created by subdividing and combining other figures and shapes (e.g., arc length, area of sectors). V.018.E. The beginning teacher understands the problem-solving process (i.e., recognizing that a mathematical problem can be solved in a variety of ways, selecting an appropriate strategy, evaluating the reasonableness of a solution). <br> V.019.C. The beginning teacher translates mathematical ideas between verbal and symbolic forms. <br> V.019.D. The beginning teacher communicates mathematical ideas using a variety of representations (e.g., numerical, verbal, graphical, pictorial, symbolic, concrete).

## Geometry TEKS

b.4. The student selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems. d.2.C. The student develops and uses formulas including distance and midpoint.
e.1.A. The student finds the area of regular polygons, and composite figures.

Background: Participants need to know the area formulas and how to connect the formulas to models of composite figures (composite of triangles, quadrilaterals and circles).

Materials: calculator

New Terms: composite figures

## Procedures:

Write the word "composite" on the overhead and brainstorm some meanings of this word. The Merriam-Webster Collegiate Dictionary, from http://www.yourdictionary.com, defines composite, when used as an adjective, as "consisting of separate interconnected parts." Remind the participants to add the new term to their glossaries.

## What are some real world examples of composite figures?

Possible answers are:
Photography: an image or scene made up of two or more original images placed side by side, overlapped, or superimposed.
Automotive: any material that consists of two or more substances bonded together for strength, such as fiberglass.
Architecture: architectural drawings, consisting of more than one room or combination of shapes.

Participants work collaboratively on problems.

1. An interior designer created a plan for the kitchen counters and an island to be located in the middle of the kitchen, as shown below. The opposite sides of the counter are parallel and the intersecting straight lines are perpendicular. The curved part of the countertop is a quarter circle. The island has parallel sides, and the curved end is a semicircle. What is the total area of the tops of the counter space and island? Use the $\pi$ key on the graphing calculator instead of $\frac{22}{7}$ or 3.14 . Round calculations to the nearest tenth.


The total areas of the tops of the counter and the island are 6232.8 sq.in.

$$
\frac{6232.8}{144} \approx 43.3 \text { sq.ft. }
$$

The counter tops and the island in the kitchen will be covered with granite. Granite costs $\$ 14.38$ per square foot. Find the cost of the countertops and island for this kitchen? Round the cost to the nearest penny.
43.3 sq.ft. $\frac{\$ 14.38}{\text { sq.ft. }}=\$ 622.654 \approx \$ 622.65$
2. $P$ is a random point on side $\overline{A Y}$ of rectangle $A R T Y$. The shaded area is what fraction of the area of the rectangle? Why?
The altitude of the triangle, $h$, is equal to the height of the rectangle. The base of the triangle is equal to the base of the rectangle. So, the area of the triangle $=\frac{1}{2}$ bh or $\frac{1}{2}$ area of the rectangle.

3. Kit and Kat are building a kite for the big kite festival. Kit has already cut his sticks for the diagonals. He wants to position $P$ so that he will have maximum kite area. He asks Kat for advice. What should Kat tell him? This problem is taken from Discovering Geometry: An Investigative Approach, Practice Your Skills, $3^{\text {rd }}$ Edition, ©2003, p.49, used with permission from Key Curriculum Press.
The location of the point of intersection of the two diagonals does not change the area, because the area of the kite is $\frac{1}{2}$
 $d_{1} d_{2}$.
4. A trapezoid has been created by combining two congruent right triangles and an isosceles triangle, as shown. Is the isosceles triangle a right triangle? How do you know? Find the area of the trapezoid two ways: first by using the trapezoid area formula, and then by finding the sum of the areas of the three triangles. This problem is taken from Discovering Geometry: An Investigative Approach, $3{ }^{\text {rd }}$ Edition, ©2003, p.420, 21, used with permission from Key Curriculum Press.


The isosceles triangle is a right triangle because the angles on either side of the right angle are complementary. If you use the trapezoid area formula, the area of the trapezoid is $\frac{1}{2}(a+b)(a+b)$. If you add the areas of the three triangles, the area of the trapezoid is $\frac{1}{2} c^{2}+a b$.
5. The rectangle and the square have equal area. The rectangle is 12 ft by 21 ft 4 in . What is the perimeter of the entire hexagon in feet?
The area of the rectangle is $256 \mathrm{ft}^{2}$. The area of the square is $256 \mathrm{ft}^{2}$. The length of the side of the square is 16 ft . The
 perimeter of the composite figure is $98 . \overline{6} \mathrm{ft}$.

Success in this activity indicates that participants are working at the Relational Level because they must solve geometric problems by selecting known properties of figures or formulas and deductive reasoning to solve problems.

## Applying Area Formulas

1. An interior designer created a plan for the kitchen counters and an island to be located in the middle of the kitchen, as shown below. The opposite sides of the counter are parallel and the intersecting straight lines are perpendicular. The curved part of the countertop is a quarter circle. The island has parallel sides, and the curved end is a semicircle. What is the total area of the tops of the counter space and island? Use the $\pi$ key on the graphing calculator instead of $\frac{22}{7}$ or

### 3.14. Round calculations to the nearest tenth.

The counter tops and the island in the kitchen will be covered with granite. Granite costs $\$ 14.38$ per square foot. Find the cost of the countertops and island for this kitchen? Round the cost to the nearest penny.

2. $P$ is a random point on side $\overline{A Y}$ of rectangle $A R T Y$. The shaded area is what fraction of the area of the rectangle? Why?

3. Kit and Kat are building a kite for the big kite festival. Kit has already cut his sticks for the diagonals. He wants to position $P$ so that he will have maximum kite area. He asks Kat for advice. What should Kat tell him? This problem is taken from Discovering Geometry: An Investigative Approach, Practice Your Skills, $3{ }^{\text {rd }}$ Edition, ©2003, p.49, used with permission from Key Curriculum Press.
4. A trapezoid has been created by combining two congruent right triangles and an isosceles triangle, as shown. Is the isosceles triangle a right triangle? How do you know? Find the area of the trapezoid two ways: first by using the trapezoid area formula, and then by finding the sum of the areas of the three triangles. This problem is taken from Discovering Geometry: An Investigative Approach, $3^{\text {rd }}$ Edition, ©2003, p.420, 21, used with permission from Key Curriculum Press.

5. The rectangle and the square have equal area. The rectangle is 12 ft by 21 ft 4 in . What is the perimeter of the entire hexagon in feet?


## What Is Surface Area?

Overview: Participants determine the surface area of a rectangular prism by counting and then finding the sum of the number of square units needed to cover each face. Group discussion deepens participants' understanding of surface area and connects the formula for surface area to the net of the solid.

Objective: TExES Mathematics Competencies
III.013.E. The beginning teacher analyzes cross-sections and nets of three-dimensional shapes.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

## Geometry TEKS

b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.D. The student uses inductive reasoning to formulate a conjecture.
d.1.B. The student uses nets to represent and construct threedimensional objects.
d.2.C. The student develops and uses formulas including distance and midpoint.
e.1.A. The student finds area of regular polygons and composite figures.

Background: No prerequisite knowledge is necessary for this activity.
Materials: centimeter grid paper, linking cubes, scissors, straightedge, tape
New Terms: lateral surface area, net, total surface area

## Procedures:

Distribute the activity sheet Intro to Nets to each participant. Tell participants that they have a two-dimensional paper figure that, when cut out and folded, will yield a solid object. Ask participants to cut out the figure, fold it, and create the solid.

## Describe the solid.

The solid is a triangular prism. It has two congruent, parallel triangular faces that are the bases of the prism, and each lateral face is a rectangle. By definition, this is a triangular prism.

Distribute the activity sheet, linking cubes, straightedge, and centimeter grid paper to participants. Ask participants to answer the question "What is surface area?"

Indicate that they will have an opportunity to share and revise their responses at the conclusion of this activity.

1. Build the solid below using linking cubes.

2. Cut rectangles from centimeter grid paper so that one rectangle will cover each face of the solid.

Participants should create and cut out six rectangles that will match the faces of the solid. They should have three pairs of congruent rectangles.
3. Cover the solid with the grid paper rectangles, matching each rectangle with the appropriate face. Tape the rectangles together.

The rectangles should completely cover the solid without overlapping.
4. Cut through enough tape connections so that the paper will unfold into a twodimensional figure that if refolded would form the original solid. This pattern is called a net. Sketch the net here.

Participants should cut only enough tape connections to unfold the paper while keeping each rectangle connected to at least one other rectangle. A possible net is shown below. Define net as a two-dimensional pattern that can be folded to form a solid.

5. Determine the number of square centimeters on each face of the net. Net is not drawn to scale.

6. Find the sum of the areas of the six faces. This represents the total surface area of the solid.

Total Surface Area $=6 \mathrm{~cm}^{2}+6 \mathrm{~cm}^{2}+8 \mathrm{~cm}^{2}+8 \mathrm{~cm}^{2}+12 \mathrm{~cm}^{2}+12 \mathrm{~cm}^{2}$
Total Surface Area $=52 \mathrm{~cm}^{2}$
Define total surface area (surface area) as the sum of the areas of the faces and curved surfaces of a solid.
Define lateral surface area of a prism as the sum of the areas of the faces excluding the area of the two parallel faces, bases, of the solid used in naming the prism. Refer back to the Intro to Nets activity. The lateral surface area of a triangular prism is the sum of the three parallelograms, rectangular faces, forming the lateral faces of the solid.
7. Complete the table below.

| Solid | Net | Total Surface Area |
| :---: | :---: | :---: | :---: | :---: |

8. Describe a method for finding the total surface area of a solid figure.

Answers will vary but should include adding the areas of all the surfaces of a solid, noting that rectangular prisms have three sets of congruent pairs of rectangular faces. The congruent pairs lie on parallel planes.

Remind participants to add the terms net, lateral surface area, and total surface area to their glossaries.

In the introductory activity Intro to Nets participants work at the Visual Level. In the activity What is Surface Area?, participants inductively determine the properties of the solid which relate to its surface area. Success with this activity indicates that participants are working at the Descriptive Level with respect to surface area of rectangular prisms.

## Intro to Nets



## What Is Surface Area?

1. Build the solid below using linking cubes.

2. Cut rectangles from centimeter grid paper so that one rectangle will cover each face of the solid.
3. Cover the solid with the grid paper rectangles, matching each rectangle with the appropriate face. Tape the rectangles together.
4. Cut through enough tape connections so that the paper will unfold into a two-dimensional figure that if refolded would form the original solid. This pattern is called a net. Sketch the net here.
5. Determine the number of square centimeters on each face of the net.
6. Find the sum of the areas of the six faces. This represents the total surface area of the solid.
7. Complete the table below.

| Solid | Net | Total Surface Area |
| :---: | :---: | :---: |
|  |  |  |
|  |  |  |

8. Describe a method for finding the total surface area of a solid figure.

## What Is Volume?

Overview: Participants determine the volume of a rectangular prism by constructing the solid from a net then counting the number of cubes needed to fill the solid. Group discussion deepens participants’ understanding of volume (the amount of space occupied by a solid measured in cubic units) and connects the formula for volume to the net of the solid.

## Objective: TExES Mathematics Competencies

III.011.B. The beginning teacher applies formulas for perimeter, area, surface area, and volume of geometric figures and shapes (e.g., polygons, pyramids, prisms, cylinders, cones, spheres) to solve problems.
III.013.E. The beginning teacher analyzes cross-sections and nets of three-dimensional shapes.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

## Geometry TEKS

b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.D. The student uses inductive reasoning to formulate a conjecture.
d.1.B. The student uses nets to represent and construct threedimensional objects.
d.2.C. The student develops and uses formulas including distance and midpoint.
e.1.A. The student finds area of regular polygons and composite figures.

Background: No prerequisite knowledge is necessary for this activity.
Materials: centimeter cubes, straightedge, centimeter grid paper, scissors, tape

## New Terms:

## Procedures:

Distribute the activity sheet, centimeter cubes, straightedge, and centimeter grid paper to participants. Ask participants to answer the question "What is volume?" Groups should record their responses to this question for future revision at the conclusion of this activity.

## What is Volume?

1. Using centimeter grid paper, build the solid (rectangular prism) from the net below.

2. Predict how many cubes will fit into the prism.

Answers may vary.
3. Check your prediction by completely filling the prism with centimeter cubes.

Participants will find that 16 cubes fill the box.
4. What is the volume of the prism, i.e., what is the total number of cubes necessary to fill the prism?
The volume of the box is $16 \mathrm{~cm}^{3}$.
5. Complete the table below.

| Net | Solid (Rectangular Prism) | Volume |
| :---: | :---: | :---: |
|  |  | $36 \mathrm{~cm}^{3}$ |
|   $A$  $A$ <br>      <br>      <br>      <br>      <br>      <br>      <br>      <br>      |  | $48 \mathrm{~cm}^{3}$ |


6. Describe a method to find the number of cubes necessary to fill a rectangular prism without actually filling it.
Answers may vary. Participants should describe finding the area of the base in square units and multiplying that number by the height of the prism in units, or the formula $V=B h$.

In groups, participants should review their responses to the question "What is volume?" Close the activity with a whole-group discussion. The following points should be addressed:

- To find the volume, in cubic units, multiply the number of cubes that completely cover one face of the prism by the number of congruent layers of cubes which completely fill the prism. Or more simply $V=l w h$.
- When the prism is viewed as a net, find three dimensions which make up the rectangular parts of the net. Two of the dimensions are used to find the area of one face; the third dimension determines the third variable in the volume formula.

Participants approach the Relational Level if they are able to describe the concept of volume directly from the net or the 3-dimensional representation. If they need to build the prism to find the volume, they are performing at the Descriptive Level. If they are able to use the formula for volume, $V=l w h$, but are unable to articulate the concept then they are probably at the Visual or Descriptive Level. Use of a formula without conceptual understanding is often misinterpreted to be at the Relational Level. Van Hiele refers to this as "level reduction". Conceptual growth cannot occur without further work at the Descriptive Level.

## What is Volume?

1. Using centimeter grid paper, build the solid (rectangular prism) from the net below.

2. Predict how many cubes will fit into the prism.
3. Check your prediction by filling the prism with centimeter cubes.
4. What is the volume of the prism, i.e., what is the total number of cubes necessary to fill the prism?
5. Complete the table below.

| Net | Solid (Rectangular Prism) | Volume |
| :---: | :---: | :---: |
|   $A$ $A$  <br>      <br>      <br>      <br>      <br>      <br>      <br>      |  |  |
|  <br>  |  |  |
|  |  |  |

6. Describe a method to find the number of cubes necessary to fill a rectangular prism without actually filling it.

## Net Perspective

Overview: Participants construct solids from nets and use nets to explore the effects on the solid's surface area and volume by changing one, two, or three dimensions of a solid.

## Objective: TExES Mathematics Competencies

II.005.C. The beginning teacher understands when a relation is a function.
II.006.G. The beginning teacher models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology.
II.004.A. The beginning teacher recognizes and extends patterns and relationships in data presented in tables, sequences, or graphs. III.011.A. The beginning teacher applies dimensional analysis to derive units and formulas in a variety of situations (e.g., rates of change of one variable with respect to another) and to find and evaluate solutions to problems.
III.011.B. The beginning teacher applies formulas for perimeter, area, and surface area, and volume of geometric figures and shapes (e.g., polygons, pyramids, prisms, cylinders, cones, spheres) to solve problems.
III.011.C. The beginning teacher recognizes the effects on length, area, or volume when the linear dimensions of plane figures or solids are changed.
III.013.E. The beginning teacher analyzes cross-sections and nets of three-dimensional shapes.
V.018.F. The beginning teacher evaluates how well a mathematical model represents a real-world situation.
V.019.A. The beginning teacher recognizes and uses multiple representations of a mathematical concept (e.g., a point and its coordinates, the area of a circle as a quadratic function of the radius, probability as the ratio of two areas, area of a plane region as a definite integral).
V.019.D. The beginning teacher communicates mathematical ideas using a variety of representations (e.g., numeric, verbal, graphical, pictorial, symbolic, concrete).

## Geometry TEKS

b.3.B. The student constructs and justifies statements about geometric figures and their properties.
d.1.B. The student uses nets to represent and construct three-dimensional objects.
e.1.D. The student finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations.
f.1. The student describes the effect on perimeter, area, and volume when length, width, or height of a three-dimensional solid is changed and applies this idea in solving problems.

Background: Participants need experience with measurement, nets, surface area, and volume.

Materials: paper, scissors, tape, rulers, centimeter grid paper (optional), centimeter cubes

## New Terms:

## Procedures:

This activity encourages participants to explore patterns in surface area and volume of rectangular prisms. Often in the middle school classroom, teachers introduce the concepts of volume and surface area of rectangular prisms by using linking cubes and asking the students to count the number of cubes to determine the volume and to count the number of faces to determine the surface area. Participants should be aware that students might develop the misconception of counting the corner cubes twice in order to find the volume of a prism.

Construct rectangular prisms with the following dimensions using centimeter cubes. Complete the table.

| Height <br> $\mathbf{c m}$ | Dimensions <br> $l \times w \times h$ | Process | Volume $_{\mathbf{c m}^{\mathbf{3}}}$ <br> 1$\quad 3 \times 2 \times 1$ |
| :---: | :---: | :---: | :---: |

1. When the dimensions of the base, the length, $l$, and the width, $w$, are fixed, how does the volume change as the height changes?
The volume increases as the height increases.
2. What function rule, in terms of the area of the base and the height, could you use to determine the volume? How does this rule relate to the model? $V=B h$, where $B$ represents the area of the base, which in this case is $3 \cdot 2=6$ and $h$ represents the height of the prism. Each time the height increases by one unit, another 6 blocks (the area of the base) are added to the solid. Thus, a prism is a stack of layers whose volume is equal to the area of the base times the height of the prism.

Construct rectangular prisms with the following dimensions using centimeter cubes.
Complete the table.

| Height <br> $\mathbf{c m}$ | Dimensions <br> $l \times w \times h$ | Process | Lateral Surface <br> Area <br> $\mathbf{c m}^{2}$ |
| :---: | :---: | :---: | :---: |
| 1 | $3 \times 2 \times 1$ | $(3+2+3+2)=2(3)(1)+2(2)(1)$ | 10 |
| 2 | $3 \times 2 \times 2$ | $(6+4+6+4)=2(3)(2)+2(2)(2)$ | 20 |
| 3 | $3 \times 2 \times 3$ | $(9+6+9+6)=2(3)(3)+2(2)(3)$ | 30 |
| 4 | $3 \times 2 \times 4$ | $(12+8+12+8)=2(3)(4)+2(2)(4)$ | 40 |
| $n$ | $3 \times 2 \times n$ | $(3 n+2 n+3 n+2 n)=2(3)(n)+2(2)(n)$ | $(2 l+2 w) h=P h$ |

3. When the dimensions of the base, the length, $l$, and the width, $w$, are fixed, how does the lateral surface area change as the height changes?
The lateral surface area increases as the height increases; every increase in height of one unit results in an increase in the lateral area of 10 square units. In this case, every increase in height of one centimeter results in an increase in the lateral area of $10 \mathrm{~cm}^{2}$.
4. What function rule, in terms of the base and the height, could you use to determine the lateral surface area? How does this rule relate to your model?
Lateral Surface Area $=P h$, where $P$ is the perimeter of the base and $h$ is the height of the prism. The lateral faces of a prism are always parallelograms (rectangles if it is a right prism) whose areas can be found by multiplying the length of the base of the parallelogram by the height of the parallelogram. The perimeter of the base of the prism is the sum of the lengths of the bases of these parallelograms.

Construct rectangular prisms with the following dimensions using centimeter cubes. Complete the table.

| Height <br> $\mathbf{c m}$ | Dimensions <br> $l \times w \times h$ | Process | Total Surface Area <br> $\mathbf{c m}^{2}$ |
| :---: | :---: | :---: | :---: |
| 1 | $3 \times 2 \times 1$ | $10+6+6$ | 22 |
| 2 | $3 \times 2 \times 2$ | $20+6+6$ | 32 |
| 3 | $3 \times 2 \times 3$ | $30+6+6$ | 42 |
| 4 | $3 \times 2 \times 4$ | $40+6+6$ | 52 |
| $n$ | $3 \times 2 \times n$ | $10 n+6+6$ | $P h+2(l w)=P h+2 B$ |

5. When the dimensions of the base, length, $l$, and width, $w$, are fixed, how does the total surface area change as the height changes?
The total surface area increases as the height increases; every increase in height of one unit results in an increase in the total surface area of 10 square units. In this
case, every increase in height of one centimeter results in an increase in the total surface area of $10 \mathrm{~cm}^{2}$.
6. What function rule, in terms of the base and the height, could you use to determine the total surface area? How does this rule relate to your model?
Total Surface Area $=P h+2 B$. The total surface area is the sum of the lateral surface area and the areas of both bases of the prism.

Distribute the activity sheet Net Perspective to each participant.
Participants are working at the Relational Level as they formulate relationships among total surface area, length, width, and height of geometric solids and analyze the effects on the solid's surface area and volume when one, two, or three dimensions of a solid are changed.

## Net Perspective

Construct rectangular prisms with the following dimensions using centimeter cubes. Complete the table.

| Height <br> $\mathbf{c m}$ | Dimensions <br> $l \times w \times h$ | Process | Volume $_{\mathbf{c m}^{3}}$ |
| :---: | :---: | :---: | :---: |
| 1 | $3 \times 2 \times 1$ |  |  |
| 2 | $3 \times 2 \times 2$ |  |  |
| 3 | $3 \times 2 \times 3$ |  |  |
| 4 | $3 \times 2 \times 4$ |  |  |
|  |  |  |  |

1. When the dimensions of the base, the length, $l$, and the width, $w$, are fixed, how does the volume change as the height changes?
2. What function rule, in terms of the area of the base and the height, could you use to determine the volume? How does this rule relate to the model?

Construct rectangular prisms with the following dimensions using centimeter cubes. Complete the table.

| Height <br> $\mathbf{c m}$ | Dimensions <br> $l \times w \times h$ | Process | Lateral <br> Surface Area <br> $\mathbf{c m}^{2}$ |
| :---: | :---: | :---: | :---: |
| 1 | $3 \times 2 \times 1$ |  |  |
| 2 | $3 \times 2 \times 2$ |  |  |
| 3 | $3 \times 2 \times 3$ |  |  |
| 4 | $3 \times 2 \times 4$ |  |  |
|  |  |  |  |

3. When the dimensions of the base, the length, $l$, and the width, $w$, are fixed, how does the lateral surface area change as the height changes?
4. What function rule, in terms of the base and the height, could you use to determine the lateral surface area? How does this rule relate to your model?

Construct rectangular prisms with the following dimensions using centimeter cubes. Complete the table.

| Height <br> $\mathbf{c m}$ | Dimensions <br> $l \times w \times h$ | Process | Total <br> Surface Area <br> $\mathbf{c m}^{2}$ |
| :---: | :---: | :---: | :---: |
| 1 | $3 \times 2 \times 1$ |  |  |
| 2 | $3 \times 2 \times 2$ |  |  |
| 3 | $3 \times 2 \times 3$ |  |  |
| 4 | $3 \times 2 \times 4$ |  |  |

5. When the dimensions of the base, the length, $l$, and the width, $w$, are fixed, how does the total surface area change as the height changes?
6. What function rule, in terms of the base and the height, could you use to determine the total surface area? How does this rule relate to your model?

## Area Proofs

Overview: Participants use area formulas and deductive reasoning in area problems.

## Objective: TExES Mathematics Competencies

III.011.B. The beginning teacher applies formulas for perimeter, area, surface area, and volume of geometric figures and shapes (e.g., polygons, pyramids, prisms, cylinders, cones, spheres) to solve problems.
III.012.A. The beginning teacher understands axiomatic systems and their components (e.g., undefined terms, defined terms, theorems, examples, counterexamples).
III.013.D. The beginning teacher computes the perimeter, area, and volume of figures and shapes created by subdividing and combining other figures and shapes (e.g., arc length, area of sectors).
V.018.A. The beginning teacher understands the nature of proof, including indirect proof, in mathematics.
V.018.B. The beginning teacher applies correct mathematical reasoning to derive valid conclusions from a set of premises. V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

## Geometry TEKS

b.1.A. The student develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.E. The student uses deductive reasoning to prove a statement. d.2.C. The student develops and uses formulas including distance and midpoint.
e.1.A. The student finds areas of regular polygons and composite figures.

Background: Participants need to be familiar with the formulas for the area of rectangles, triangles, parallelograms, trapezoids, and kites.

Materials: colored markers, easel paper

## New Terms:

## Procedures:

Allow time for participants to work independently on proofs. Then assign one proof to each group to be recorded on easel paper. As time permits, participants can walk the
room, observing similarities and differences in the proofs they have done independently and the proofs on display. Proofs can be done informally in paragraph form.

1. Given: $B$ is the midpoint of $\overline{A C}$

$$
\overline{E D} \square \overline{A C}
$$

Prove: Area of $\triangle A B E=$ Area of $\triangle B C D$
Draw $h_{1}$, the height of $\triangle E A B$, and

$h_{2}$, the height of $\triangle D B C$.
$h_{1}=h_{2}$
$(\overline{E D} \square \overline{A C})$
Area of $\triangle A B E=\frac{1}{2} h_{1} \cdot A B$
Area of $\triangle B C D=\frac{1}{2} h_{2} \cdot B C$
(Area of triangle $=\frac{1}{2}$ base $\cdot$ height $)$

$$
\begin{aligned}
& A B=B C \\
& \frac{1}{2} h_{1} \cdot A B=\frac{1}{2} h_{2} \cdot B C
\end{aligned}
$$

$$
\text { ( } B \text { is the midpoint of } \overline{A C} \text { ) }
$$

(Substitution)

Area of $\triangle A B E=$ Area of $\triangle B C D$
(Substitution)
2. Given: Trapezoid $A B C D$ with diagonals intersecting at $P$

Prove: Area of $\triangle A P D=A$ rea of $\triangle B P C$


Area of $\triangle A D C=$ Area of $\triangle B C D$
Area of $\triangle A D C-$ Area of $\triangle D P C=$ Area of $\triangle B C D$ - Area of $\triangle D P C$

Area $\triangle A P D=$ Area $\triangle B P C$.
( $\triangle A D C$ and $\triangle B C D$ have the same base and height)
(Subtraction)
(Substitution of composite parts)
3. Given: $\overline{A D}$ is a median of $\triangle \mathrm{ABC}$

Prove: Area $\triangle C A D=$ Area $\triangle B A D$

Draw $\overline{A E}$, the altitude to base $\overline{C B}$ of $\triangle A B C$.


Area of $\triangle C A D=\frac{1}{2} A E \cdot C D$
(Area formula)
Area of $\triangle B A D=\frac{1}{2} A E \cdot D B$
(Area formula)
$C D=D B$
(Definition of median)
Area of $\triangle C A D=$ Area of $\triangle B A D$
(Substitution)
4. Given: Diagonals $\overline{D B}$ and $\overline{A C}$ of quadrilateral $A B C D$ are perpendicular.
Prove: Area of $A B C D=\frac{1}{2} D B \cdot A C$


Area of quadrilateral $A B C D=$
(Composite parts)
Area of $\triangle A D C+$ Area of $\triangle A B C$
Area of $\triangle A D C=\frac{1}{2} D E \cdot A C$
(Area formula)
Area of $\triangle A B C=\frac{1}{2} B E \cdot A C$

Area of quadrilateral $A B C D=$
(Substitution)
$\frac{1}{2} D E \cdot A C+\frac{1}{2} B E \cdot A C$
$\frac{1}{2} D E \cdot A C+\frac{1}{2} B E \cdot A C=\frac{1}{2} \cdot A C(D E+B E)$ (Distributive property)
$\frac{1}{2} \cdot A C(D E+B E)=\frac{1}{2} A C \cdot D B$
(Substitution of composite parts)
5. Given: $\triangle M Q R$ with medians $\overline{R S}$ and $\overline{M T}$ intersecting at $P$
Prove: Area $\triangle P M S=$ Area $\triangle P R T$


Area of $\triangle T M Q=\frac{1}{2}\left(M Q \cdot \frac{1}{2} h\right)$
(Area formula)
Area of $\triangle S R Q=\frac{1}{2}\left(\frac{1}{2} \cdot M Q\right) h \quad$ (Area formula)
Area of $\triangle T M Q=$ Area of $\triangle S R Q \quad$ (Substitution)
Area of $\triangle T M Q$ - Area of quadrilateral TPSQ = (Subtraction)
Area of $\triangle S R Q$ - Area of quadrilateral TPSQ

Area of $\triangle P M S=$ Area of $\triangle P R T$
(Substitution of composite parts)
Success in this activity indicates that participants are working at or approaching the Deductive Level. Whereas at the Relational Level deductive argument is informal, proof at the Deductive Level requires the use of an axiomatic system with formal definitions and postulates.

## Area Proofs

Use area formulas and the properties of triangles and quadrilaterals to prove each of the following area relationships. Use an informal presentation, such as a flow chart proof or paragraph proof, to demonstrate your deductive reasoning.

1. Given: $B$ is the midpoint of $\overline{A C}$

$$
\overline{E D} \square \overline{A C}
$$

Prove: Area of $\triangle A B E=$ Area of $\triangle B C D$

2. Given: Trapezoid $A B C D$ with diagonals intersecting at $P$

Prove: Area of $\triangle A P D=$ Area of $\triangle B P C$

3. Given: $\overline{A D}$ is a median of $\triangle A B C$

Prove: Area of $\triangle C A D=$ Area of $\triangle B A D$

4. Given: Diagonals $\overline{D B}$ and $\overline{A C}$ of quadrilateral $A B C D$ are perpendicular.

Prove: Area of $A B C D=\frac{1}{2} \overline{D B} \cdot \overline{A C}$

5. Given: $\triangle M Q R$ with medians
$\overline{R S}$ and $\overline{M T}$ intersecting at $P$
Prove: Area of $\triangle P M S=$ Area of $\triangle P R T$


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## Unit 6 - Pythagoras

## Sides of Squares

Overview: Participants discover the Pythagorean Theorem inductively by finding the areas of squares.

Objective: TExES Mathematics Competencies
II.004.A. The beginning teacher recognizes and extends patterns and relationships in data presented in tables, sequences, or graphs. III.012.A.The beginning teacher understands axiomatic systems and their components (e.g., undefined terms, defined terms, theorems, examples, counterexamples).
III.013.D. The beginning teacher computes the perimeter, area, and volume of figures and shapes created by subdividing and combining other figures and shapes (e.g., arc length, area of sectors). III.014.E. The beginning teacher applies concepts and properties of slope, midpoint, parallelism, perpendicularity, and distance to explore properties of geometric figures and solve problems in the coordinate plane.
V.018.A. The beginning teacher understands the nature of proof, including indirect proof, in mathematics.
V.018.C. The beginning teacher uses inductive reasoning to make conjectures and uses deductive methods to evaluate the validity of conjectures.

## Geometry TEKS

b.1.A. The student develops an awareness of the structure of a mathematical system, connecting definition, postulates, logical reasoning, and theorems.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.C. The student demonstrates what it means to prove mathematically that statements are true.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
d.2.A. The student uses one- and two-dimensional coordinate systems to represent point, lines, line segments, and figures.
d.2.B. The student uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons.
e.1.A. The student finds areas of regular polygons and composite figures.
e.1.C. The student develops, extends, and uses the Pythagorean Theorem.

Background: Participants need the ability to apply the formulas for the area of squares and right triangles.

Materials: centimeter grid paper

## New Terms:

## Procedures:

Conduct a brief discussion with participants regarding the difficulties embedded in grasping the concept of the Pythagorean Theorem when it is presented as a proven fact without prior conceptual exploration.

Start with the area problem presented in the area unit, which is a proof of the Pythagorean Theorem. Participants return to Unit 5, Applying Area Formulas.

A trapezoid was created by combining two congruent right triangles and an isosceles triangle, as shown. Is the isosceles triangle a right triangle? How do you know? Find the area of the trapezoid two ways: first by using the trapezoid area formula, and then by finding the sum of the areas of the three triangles.

This problem is taken from Discovering Geometry: An Investigative Approach, $3^{\text {rd }}$ Edition, ©2003, p.419, 17, used with permissions from Key Curriculum Press.


The isosceles triangle is a right triangle because $\angle 1$ and $\angle 2$ are complementary and $\angle 1$, $\angle 2$ and $\angle 3$ form a straight angle, therefore $\mathrm{m} \angle 3=90^{\circ}$. If you use the trapezoid area formula, the area of the trapezoid is $\frac{1}{2}(a+b)(a+b)$. If you add the areas of the three triangles, the area of the trapezoid is $\frac{1}{2} c^{2}+a b$.
Equating the two and simplifying:

$$
\begin{array}{ll}
\frac{1}{2}(a+b)(a+b)=\frac{1}{2} c^{2}+a b . \\
\frac{1}{2}\left(a^{2}+2 a b+b^{2}\right) & =\frac{1}{2} c^{2}+a b . \\
\frac{1}{2} a^{2}+a b+\frac{1}{2} b^{2} & =\frac{1}{2} c^{2}+a b . \\
\frac{1}{2} a^{2}+\frac{1}{2} b^{2} & =\frac{1}{2} c^{2} . \\
a^{2}+b^{2} & =c^{2} .
\end{array}
$$

The above proof is attributed to President Garfield. See Pappas (1986).
Discussion questions:
What can be understood conceptually about the Pythagorean Theorem from the proof that emerged from the area problem?
The proof provides no conceptual understanding of the meanings of $a^{2}, b^{2}$, and $c^{2}$ as geometric entities.

What van Hiele level is represented by the area problem with respect to the Pythagorean Theorem?
Success with this problem indicates that a person is working at the Deductive Level. The relationship is abstract, derived algebraically, and without connection to the geometric or contextual meanings of $a^{2}, b^{2}$, and $c^{2}$.

How is the Pythagorean Theorem usually presented?
In textbooks, often a figure of a right triangle with squares drawn on its sides is presented with the relationship $a^{2}+b^{2}=c^{2}$. Students are expected to accept the diagram as a "proof" and then memorize and apply the result. This may be considered to be the final phase of concept development in the van Hiele model. The guided discovery has been omitted.

What concepts must be in place for concept development of the Pythagorean Theorem?
Students must be able to find the side length of a square given its area.
Distribute the activity sheets. Arrange participants in groups of three to complete Part 1 with each participant completing one of the three grids A, B, or C. Participants then complete Part 2 together. Continue working to complete Part 2 prior to whole class discussion.

## Part 1

For grids A, B and C, complete the following steps for each line segment. Let the slope of each segment be $\frac{a}{b}$.

- Write the slope, in unsimplified form, next to the segment. In Grid A, $a=1$; in Grid B, $a=2$; in Grid C, $a=3$.
- Using your knowledge of parallel and perpendicular lines, build a square that is on the upper left side of each segment.
- Divide each square into a composite of right triangles and squares by drawing segments in from the vertices, along the horizontal or vertical grid lines.
- Find the area of the original square from the sum of the areas of the composite figures.
- Find the length of each original segment from the area of its square.

Grid A $\quad$ Slope $=\frac{1}{b}$


Slope $=\frac{1}{1}$
Area of square

$$
\begin{aligned}
& =4 \cdot \frac{1}{2} \\
& =2
\end{aligned}
$$

$$
\text { length }=\sqrt{2}
$$

Slope $=\frac{1}{3}$
Area of square
$=\left(4 \cdot \frac{3}{2}\right)+4$
$=10$

Slope $=\frac{1}{4}$
Area of square
$=(4 \cdot 2)+9$
$=17$
length $=\sqrt{17} \approx 4.12$

Grid B $\quad$ Slope $=\frac{2}{b}$


$$
\begin{array}{cll}
\text { Slope }=\frac{2}{2} & \text { Slope }=\frac{2}{3} & \text { Slope }=\frac{2}{4} \\
\text { Area of square } & \text { Area of square } & \text { Area of square }=4(4)+4 \\
=4(2) & =4(3)+1 & \text { l ength }=\sqrt{20} \approx 4.47 \\
=8 & =13 & \\
\text { length }=\sqrt{8} \approx 2.83 & \text { length }=\sqrt{13} \approx 3.61 &
\end{array}
$$

Grid C $\quad$ Slope $=\frac{3}{b}$


$$
\text { Slope }=\frac{3}{3}
$$

Area of square

$$
\begin{aligned}
& =4 \cdot \frac{9}{2} \\
& =18
\end{aligned}
$$

length $=\sqrt{18} \approx 4.24$

Slope $=\frac{3}{4}$
Area of square

$$
\begin{aligned}
& =(4 \cdot 6)+1 \\
& =25
\end{aligned}
$$

length $=\sqrt{25}=5$

## Part 2

Complete the following table. Write the length of the original segment in unsimplified radical form (i.e., $\sqrt{2}$, rather than $1.14 \ldots$ )

|  | Slope | Area of Original Square | Length of Original <br> Segment |
| :--- | :---: | :---: | :---: |
| Grid A | $\frac{1}{1}$ | 2 | $\sqrt{2}$ |


| Slope $=\frac{1}{b}$ | $\frac{1}{2}$ | 5 | $\sqrt{5}$ |
| :--- | :---: | :---: | :---: |
|  | $\frac{1}{3}$ | 10 | $\sqrt{10}$ |
|  | $\frac{1}{4}$ | 17 | $\sqrt{17}$ |
| Grid B | $\frac{2}{2}$ | 8 | $\sqrt{8}$ |
|  | $\frac{2}{3}$ | 13 | $\sqrt{13}$ |
|  | $\frac{2}{4}$ | 20 | $\sqrt{20}$ |
| Grid C | $\frac{3}{3}$ | 18 | $\sqrt{18}$ |
| Slope $=\frac{3}{b}$ | $\frac{3}{4}$ | 25 | $\sqrt{25}=5$ |

In your group, using the grid figures and the table, discuss and determine relationships among slope numbers, area and segment length. Be prepared to share your findings in whole class discussion.

When most groups have completed Parts 1 and 2, call on a participant to record relationships on the board or easel paper.

The relationships that must emerge are:

- The area of the square is equal to the sum of the squares of the numbers in the unsimplified slope ratio. To generalize, if the segment's slope (in unsimplified form) is $\frac{a}{b}$, then the area of the square on the segment is $a^{2}+b^{2}$.
- The length of the segment is given by $\sqrt{a^{2}+b^{2}}$.

Other relationships that may emerge:

- In the grid figures, when the slopes are of the form $\frac{a}{a}$, there is no middle square, and the area is given by $2 a^{2}$.
- When the numbers in the slope ratio differ by 1 , the area of the middle square is $1^{2}$.
- When the numbers in the slope ratio differ by 2 , the area of the middle square is $2^{2}$.
- When the numbers in the slope ratio differ by 3 , the area of the middle square is $3^{2}$.
- The base and the height of each of the four congruent triangles are always the two numbers in the unsimplified slope ratio.
- To generalize, if slope $=\frac{a}{b}$ (in unsimplified form), the area of the middle square is $(a-b)^{2}$; the area of each of the four congruent triangles is $\frac{a b}{2}$.
Total area $=(a-b)^{2}+4\left(\frac{a b}{2}\right)$

$$
\begin{aligned}
& =a^{2}-2 a b+b^{2}+2 a b \\
& =a^{2}+b^{2} .
\end{aligned}
$$

## Part 3

Note: The activity sheet does not include Part 3. This part is trainer-directed and makes the connection to the right triangle with squares on the legs and hypotenuse. Using the vertical and horizontal grid lines, draw the right triangle on an original side of one of the squares. Continue the discussion as follows:

What is the geometric meaning of $\boldsymbol{a}^{\mathbf{2}}$ or $\boldsymbol{b}^{\mathbf{2}}$ ?
If $a($ or $b)$ is a length measurement, then $a^{2}$ (or $b^{2}$ ) is the area of a square with side measurement $a$ (or $b$ ).

Participants draw and shade in the right triangle on one of the sides of the square using the side of the square as the hypotenuse. Show its legs, which are $a$ and $b$, as vertical and horizontal segments. Then draw in the squares that represent the areas $a^{2}$ and $b^{2}$.



When most participants have completed drawing the squares with horizontal and vertical sides, make sure that they also compute the areas and connect the symbolic statement of the Pythagorean Theorem, $a^{2}+b^{2}=c^{2}$, to the geometric, pictorial representations in each figure.

Ask the following question and allow groups a few minutes to share their answers. How would you express the Pythagorean Theorem in words now that you have a symbolic and pictorial representation?
Considering a right triangle with squares constructed on the legs and the hypotenuse, the area of the largest square, the one on the hypotenuse, is equal to the sum of the areas of the squares on the two legs.


Make a connection to triangle congruence using the correspondence of hypotenuse and leg, or hypotenuse and acute angle between right triangles.

Are two right triangles congruent if they have congruent hypotenuses and one of the legs on one triangle is congruent to one of the legs on the other triangle?
If the lengths of the hypotenuse and one leg of a right triangle are known, then, using the Pythagorean Theorem, the other leg is determined. Since the two triangles now have three congruent sides congruent to three congruent sides, SSS, the triangles are congruent. This is referred to as the Hypotenuse-Leg Triangle Congruence Theorem (HL).

Are two right triangles congruent if they have congruent hypotenuses and one of the acute angles on one triangle is congruent to one of the acute angles on the other triangle?
If the measures of two angles are known, namely the right angle and the acute angle, then the third angle is known. Since one triangle has three angles and the hypotenuse congruent to three angles and the hypotenuse on the other triangle, ASA, the triangles must be congruent. This is referred to as the Hypotenuse-Angle Triangle Congruence Theorem (HA).

Success in Part 1 of the activity indicates that participants are working at the van Hiele Descriptive Level because the figures are analyzed in terms of their components. The relationships between the component parts are identified.

In Part 2, success indicates that participants are working at the Relational Level. The activity requires them to formulate and use definitions and to draw conclusions from informal arguments.

In Part 3, success indicates that participants are working at the Relational Level, because algebraic relationships are connected to their geometric counterparts deductively.

In the next activity, participants establish the relationship between the areas of the squares on the two shorter sides and the area on the longest side of an acute and an obtuse triangle.

## Sides of Squares

## Part 1

For grids A, B and C, complete the following steps for each line segment.
Let the slope of each segment be $\frac{a}{b}$.

- Write the slope, in unsimplified form, next to the segment. In Grid A, $a=$ 1 ; in Grid B, $a=2$; in Grid C, $a=3$.
- Using your knowledge of parallel and perpendicular lines, build a square that is on the upper left side of each segment.
- Divide each square into a composite of right triangles and squares by drawing segments in from the vertices, along the horizontal or vertical grid lines.
- Find the area of the original square from the sum of the areas of the composite figures.
- Find the length of each original segment from the area of its square.

$$
\text { Grid } \mathrm{A} \quad \text { Slope }=\frac{1}{b}
$$



Grid B $\quad$ Slope $=\frac{2}{b}$

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Grid C $\quad$ Slope $=\frac{3}{b}$

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
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## Part 2

Complete the following table. Write the length of each of the original segments in unsimplified radical form (i.e., $\sqrt{2}$, rather than 1.14...)

|  | Slope | Area of Original <br> Square | Length of Original <br> Segment |
| :--- | :--- | :---: | :---: |
| Grid A <br> Slope $=\frac{1}{b}$ |  |  |  |
|  |  |  |  |
|  |  |  |  |
| Grid B <br> Slope $=\frac{2}{b}$ |  |  |  |
|  |  |  |  |
|  |  |  |  |
| Grid C |  |  |  |
| Slope $=\frac{3}{b}$ |  |  |  |

In your group, using the grid figures and the table, discuss and determine relationships among slope numbers, area and segment length. Be prepared to share your findings in whole class discussion.

## Part 3

Conclusions

## Squares on the Sides of Acute or Obtuse Triangles

Overview: | Participants test the validity of the Pythagorean Theorem inductively |
| :--- |
| by finding the areas of squares on the sides of acute and obtuse |
| triangles. |
| Objective: |
| TExES Mathematics Competencies |
| V.018.A. The beginning teacher understands the nature of proof, |
| including indirect proof, in mathematics. |
| V.018.C. The beginning teacher uses inductive reasoning to make |
| conjectures and uses deductive methods to evaluate the validity of |
| conjectures. |
| V.018.D. The beginning teacher uses formal and informal reasoning to |
| justify mathematical ideas. |
| Geometry TEKS |
| b.3.C. The student demonstrates what it means to prove |
| mathematically that statements are true. |
| d.2.A. The student uses one- and two-dimensional coordinate systems |
| to represent point, lines, line segments, and figures. |
| e.1.C. The student develops, extends, and uses the Pythagorean |

Theorem.

Background: $\quad$| Participants need to be able to find side lengths of squares when given |
| :--- |
| their area. |

Materials: $\quad$| centimeter grid paper, centimeter ruler |
| :--- |

## New Terms:

## Procedures:

Distribute centimeter grid paper to participants.

1. Draw a scalene acute triangle in the center of the grid paper, with one of its sides along the horizontal or vertical lines. Label the longest side of the triangle $c$ and the other two sides $a$ and $b$. Make sure each vertex is at the intersection of a horizontal and a vertical grid line.

Draw the squares on each of the three sides of the triangle. Using the same dividing process as in the previous activity, find the area of each of the squares, $a^{2}, b^{2}$, and $c^{2}$.

Determine the relationship among $a^{2}, b^{2}$, and $c^{2}$ for an acute triangle.

An example is shown below.


If participants find the process confusing, it may help to cover over most of the acute triangle, leaving visible only the segment whose square is being constructed.
2. Draw a scalene obtuse triangle in the center of the grid paper with one of its sides along the horizontal or vertical lines. Label the longest side $c$, and the other two sides $a$ and $b$. Make sure each vertex is at the intersection of a horizontal and a vertical grid line.

As with the acute triangle, draw the squares on the three sides, and find the areas of the three squares, $a^{2}, b^{2}$, and $c^{2}$.

Determine the relationship among $a^{2}, b^{2}$, and $c^{2}$ for an obtuse triangle.
An example is shown.

$$
\begin{aligned}
a^{2}+b^{2} & =2+10=12 . \\
c^{2} & =16 . \\
12 & <16 . \\
a^{2}+b^{2} & <c^{2} .
\end{aligned}
$$



Does the relationship $a^{2}+b^{2}=c^{2}$ still apply to acute triangles?
No. In acute triangles, if $c$ is the longest side, $a^{2}+b^{2}>c^{2}$.
Does the relationship $a^{2}+b^{2}=c^{2}$ still apply to obtuse triangles?
No. In obtuse triangles, if $c$ is the longest side, $a^{2}+b^{2}<c^{2}$.
The following illustration provides additional support for this result. Consider the three triangles, $\triangle A B_{1} C, \Delta A B_{2} C$, and $\Delta A B_{3} C$. Two sides in each of the three triangles are congruent. $\overline{A C}$ is common to all three triangles, and $B_{1} C=B_{2} C=B_{3} C$, since all are radii of circle $C$.

$\Delta A B_{I} C$ is a right triangle, with right angle at the center of circle $C$.
Therefore, $(A C)^{2}+\left(B_{l} C\right)^{2}=\left(B_{l} A\right)^{2}$.
$\triangle A B_{2} C$ is an acute triangle.
Two sides of $\Delta A B_{2} C$ are congruent to two sides of $\Delta A B_{1} C\left(B_{2} C=B_{1} C ; A C=A C\right)$. But $B_{1} A>B_{2} A$, therefore $(A C)^{2}+\left(B_{2} C\right)^{2}>\left(B_{2} A\right)^{2}$.
$\triangle A B_{3} C$ is an obtuse triangle.
Two sides of $\Delta A B_{3} C$ are congruent to two sides of $\Delta A B_{1} C\left(B_{3} C=B_{1} C ; A C=A C\right)$.
But $B_{1} A<B_{3} A$, therefore $(A C)^{2}+\left(B_{3} C\right)^{2}<\left(B_{3} A\right)^{2}$.
Success in the activity indicates that participants are working at the Relational Level, because they provide informal, deductive arguments for the relationship of the squares of lengths of sides in a triangle. Moreover, properties of different triangles are interrelated.

## Squares on the Sides of Acute or Obtuse Triangles

1. Draw a scalene acute triangle in the center of the grid paper, with one of its sides along the horizontal or vertical lines. Label the longest side of the triangle $c$ and the other two sides $a$ and $b$. Make sure each vertex is at the intersection of a horizontal and a vertical grid line.

Draw the squares on each of the three sides of the triangle. Using the same dividing process as in the previous activity, find the area of each of the squares, $a^{2}, b^{2}$, and $c^{2}$.

Determine the relationship among $a^{2}, b^{2}$, and $c^{2}$ for an acute triangle.
2. Draw a scalene obtuse triangle in the center of the grid paper with one of its sides along the horizontal or vertical lines. Label the longest side $c$, and the other two sides $a$ and $b$. Make sure each vertex is at the intersection of a horizontal and a vertical grid line.

As with the acute triangle, draw the squares on the three sides, and find the areas of the three squares, $a^{2}, b^{2}$, and $c^{2}$.

Determine the relationship among $a^{2}, b^{2}$, and $c^{2}$ for an obtuse triangle.

## Applying Pythagoras, Part I

Overview: Participants apply the Pythagorean Theorem to a variety of problems.

## Objective: TExES Mathematics Competencies

II.006.G. The beginning teacher models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology.
III.011.D. The beginning teacher applies the Pythagorean Theorem, proportional reasoning, and right triangle trigonometry to solve measurement problems.
III.013.D. The beginning teacher computes the perimeter, area, and volume of figures and shapes created by subdividing and combining other figures and shapes (e.g., arc length, area of sectors).
V.019.C. The beginning teacher translates mathematical ideas between verbal and symbolic forms.

## Geometry TEKS

b.4. The student uses a variety of representations to describe geometric relationships and solve problems.
c.3. The student identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-6090 ) and triangles whose sides are Pythagorean triples.
e.1.C. The student develops, extends, and uses the Pythagorean Theorem.

Background: Participants should be able to apply the Pythagorean Theorem.
Materials: calculator

## New Terms:

## Procedures:

Distribute the activity sheet. Participants work independently or in small groups to complete the six problems.

1. Find the area in square units of trapezoid $B C D E$ if the length of $\overline{A C}$ is 20 units, the length of $\overline{D C}$ is 12 units, and the length of $\overline{B E}$ is 3 units.


Applying the Pythagorean Theorem on $\triangle A C D$, $A D=16$ units.
$\triangle A B E \sim \triangle A C D . B E$ is one-fourth of $C D$; therefore $A E$ is one-fourth of $A D . A E=4$ units.
$D E=A D-A E=12$ units.
Area of trapezoid BCDE
$=\frac{(3+12)}{2} 12=90$ square units.
2. The diagonals of a rhombus are 6 inches and 8 inches. What is the perimeter, in inches, of the rhombus?


Applying the Pythagorean Theorem, the length of each side of the rhombus is 5 inches.
The perimeter is 20 inches.
3. What is the area in square units of the quadrilateral $F G H J$ ?


The area of $\triangle F G H$ is 344 square units. Applying the Pythagorean Theorem to $\Delta F G H, F H=40$ units.
Applying the Pythagorean Theorem to $\triangle F H J, F J=96$ units.
The area of $\triangle F H J$ is 1920 square units. The area of quadrilateral FGHJ is 2304 square units.
4. Find the area of the isosceles trapezoid in square feet. Express your answer in simplest radical form.


Draw the perpendicular segments from the ends of the short base to the long base, forming right triangles. Applying the Pythagorean Theorem, the height of the triangles and the height of the trapezoid is $2 \sqrt{10} \mathrm{ft}$.
The area of the trapezoid is $30 \sqrt{10} \mathrm{ft}^{2}$.
5. What is the length of $\overline{A G}$ ?


Applying the Pythagorean Theorem on
$\triangle A B C, A C=\sqrt{5}$ units.
Applying the Pythagorean Theorem on
$\triangle A C D, A D=\sqrt{6}$ units.
Continuing in this fashion
$A G=\sqrt{9}=3$ units.
6. A traveler drove 18 miles north, then 11 miles west, then 6 miles south, and then 6 miles east. In miles, how far "as the crow flies" was the traveler from his original starting point?


Sketch the route as shown. The legs of the right triangle measure 12 miles and 5 miles. Using the Pythagorean Theorem, the hypotenuse measures 13 miles. The traveler was 13 miles from his original starting point.

Success in this activity indicates that participants are working at the van Hiele Relational Level, because they use properties of different figures along with formulas to solve problems.

## Applying Pythagoras, Part I

1. Find the area in square units of trapezoid $B C D E$ if the length of $\overline{A C}$ is 20 units, the length of $\overline{D C}$ is 12 units, and the length of $\overline{B E}$ is 3 units.

2. The diagonals of a rhombus are 6 inches and 8 inches. What is the perimeter, in inches, of the rhombus?
3. What is the area in square units of the quadrilateral $F G H J$ ?

4. Find the area of the isosceles trapezoid in square feet. Express your answer in simplest radical form.

$18^{\prime}$
5. What is the length of $\overline{A G}$ ?

6. A traveler drove 18 miles north, then 11 miles west, then 6 miles south, and then 6 miles east. In miles, how far "as the crow flies" was the traveler from his original starting point?

## Pythagorean Triples

Overview: Participants discover relationships that will enable them to quickly recall Pythagorean triples.

## Objective: TexES Mathematics Competencies

II.004.A. The beginning teacher recognizes and extends patterns and relationships in data presented in tables, sequences, or graphs.
II.006.G. The beginning teacher models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology.

## Geometry TEKS

b.3.C. The student demonstrates what it means to prove mathematically that statements are true.
c.3. The student identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-6090) and triangles whose sides are Pythagorean triples. e.1.C. The student develops, extends, and uses the Pythagorean Theorem.
f.1. The student develops, applies, and justifies triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples.

Background: Participants should be able to apply the Pythagorean Theorem.
Materials: transparencies of the tables for the activity, calculator
New Terms: Pythagorean triple

## Procedures:

Introduce the activity by referring back to the problem solving section of the last activity. Problems 1, 2, 3, and 6 include right triangles whose sides are all natural numbers. When three natural numbers satisfy the Pythagorean Theorem, they are called Pythagorean triples or triplets. Remind participants to add the new term Pythagorean triple to their glossaries.

The triples in these problems are in ratios of 3:4:5 and 5:12:13. Multiples of these triples appear in 1,2 , and 3 of the current activity sheet.

Interesting patterns and generalizations exist in certain groups of triples. The lengths of the legs of a right triangle are $a$ and $b$, and the length of the hypotenuse is $c$. In the first table, $a$ is odd; in the second table, $a$ is even.

Participants examine patterns within triples to enable them to easily remember and generate other Pythagorean triples.

Distribute the activity sheet. The activity requires instructor-participant interaction throughout.

Table 1: $a$ is odd.
Work with your group to complete the three rows where $a=3,5$, and 7 . Be prepared to share in whole class discussion.

Participants complete the row beginning with $a=3$. The columns labeled $a^{2}, b^{2}$, and $c^{2}$ are for checking that the numbers chosen for $a, b$, and $c$ are indeed Pythagorean triples.

| $\boldsymbol{a}$ | $\boldsymbol{b}$ |  | $\boldsymbol{c}$ | $\boldsymbol{a}^{\mathbf{2}}+\boldsymbol{b}^{\mathbf{2}}$ | $=\boldsymbol{c}^{\mathbf{2}}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |  |  |
| 3 | 4 |  | 5 | 9 | 16 | 25 |
| 5 | 12 |  | 13 | 25 | 144 | 169 |
| 7 | 24 |  | 25 | 49 | 576 | 625 |

Cover the columns for $b^{2}$ and $c^{2}$.
What patterns can be seen in this part of the table?

| $\boldsymbol{a}$ | $\boldsymbol{b}$ |  | $\boldsymbol{c}$ | $\boldsymbol{a}^{\mathbf{2}}$ |
| :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |
| 3 | 4 |  | 5 | 9 |
| 5 | 12 |  | 13 | 25 |
| 7 | 24 |  | 25 | 49 |

Several patterns may be discerned, but the following two patterns must emerge:
$b+c=a^{2}$ and
$b$ and $c$ differ by 1 , or $c-b=1$.
Using this information, if $a=9$, then $a^{2}=81$. What are the values for $b$ and $c$ ?
$b+c=81$, and $c-b=1$.
$b=40, c=41$.

## How may one find $\boldsymbol{b}$ and $\boldsymbol{c}$ quickly?

Find one half of $a^{2}$, which is 40.5 . Subtract 0.5 from 40.5 to find $b$, which is 40 . Add 0.5 to 40 to find $c$, which is 41 .
Alternately, subtract 1 from $a^{2}$ to get 80 . Divide 80 by 2 to get $b$, which is 40 . Add 1 to 40 to get $c, 41$.

Write the following numbers in the table. Participants find the missing numbers using the inductive rule determined above.

| $\boldsymbol{a}$ | $\boldsymbol{b}$ |  | $\boldsymbol{c}$ | $\boldsymbol{a}^{\mathbf{2}}$ |
| :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |
| 3 | 4 |  | 5 | 9 |
| 5 | 12 |  | 13 | 25 |
| 7 | 24 |  | 25 | 49 |
| 9 |  |  |  |  |
| 11 |  |  |  |  |
|  | 112 |  |  |  |
|  |  |  | 221 |  |

Do the numbers 9, 40, and 41 form a Pythagorean triple? Test the numbers in the relationship $a^{2}+b^{2}=\boldsymbol{c}^{2}$.
The numbers 9,40 , and 41 form a Pythagorean triple, because they satisfy $a^{2}+b^{2}=c^{2}$. $9^{2}+40^{2}=81+1600=1681=41^{2}$.

What is the triple for $a=11$ ? Explain.
The numbers 11,60 , and 61 form a Pythagorean triple.
If $a=11$, then $a^{2}=121$.
We also know $b+c=121$.
To find $b$ and $c$ quickly, find half of 121 , which is 60.5 .
Then subtract 0.5 from 60.5 to get $b$, which is 60 .
Add 0.5 to 60.5 to get $c$, which is 61 .
Note that $60+61=121$.
To check: $11^{2}+60^{2}=121+3600=3721=61^{2}$
Thus, $11^{2}+60^{2}=61^{2}$, which satisfies the Pythagorean Theorem.
What is the triple for $\boldsymbol{b}=112$ ? Explain.
The numbers 15,112 , and 113 form a Pythagorean triple.
If $b=112$, then $c=113$.
We know $a^{2}=b+c$.
Substituting 112 and 113 for $b$ and $c, a^{2}=b+c=112+113=225$.
Therefore, $a=15$.
Check: $a^{2}+b^{2}=15^{2}+112^{2}=225+12,544=12,769$.
$c^{2}=113^{2}=12,769$.
The answers for $a^{2}+b^{2}$ and $c^{2}$ are the same, 12,769 .
What is the triple for $\boldsymbol{c}=\mathbf{2 2 1}$ ? Explain.
The numbers 21, 220, and 221 form a Pythagorean triple.
If $c=221$, then $b=220$.
Since $a^{2}=b+c, a^{2}=221+220=441$.
We now know that $a=21$.
Check: $a^{2}+b^{2}=21^{2}+220^{2}=441+48,400=48,841$.
$c^{2}=221^{2}=48,841$.
Thus, $a^{2}+b^{2}$ is the same as $c^{2}$. Both are 48,841 .

What is the triple in terms of $a$ ?

| $\boldsymbol{a}$ | $\boldsymbol{b}$ |  | $\boldsymbol{c}$ | $\boldsymbol{a}^{\mathbf{2}}$ |
| :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |
| 3 | 4 |  | 5 | 9 |
| 5 | 12 |  | 13 | 25 |
| 7 | 24 |  | 25 | 49 |
| 9 | 40 |  | 41 | 81 |
| 11 | 60 |  | 61 | 121 |
| 15 | 112 |  | 113 | 225 |
| 21 | 220 |  | 221 | 441 |
| $a$ | $\frac{1}{2}\left(a^{2}-1\right)$ |  | $\frac{1}{2}\left(a^{2}+1\right)$ | $a^{2}$ |

Confirm the identity $a^{2}+\left(\frac{1}{2}\left(a^{2}-1\right)\right)^{2}=\left(\frac{1}{2}\left(a^{2}+1\right)\right)^{2}$.

$$
\begin{aligned}
& a^{2}+\left(\frac{1}{2}\left(a^{2}-1\right)\right)^{2} \\
& =a^{2}+\frac{1}{4}\left(a^{4}-2 a^{2}+1\right) \\
& =\frac{1}{4} a^{4}+\frac{1}{2} a^{2}+\frac{1}{4} \\
& =\left(\frac{1}{2}\left(a^{2}+1\right)\right)^{2}
\end{aligned}
$$

Insert $a=1$ in the first row of the table.
What is the triple for $a=1$, the first odd whole number? Explain.
The triple is $1,0,1$.
$a^{2}=1$ and
$b+c=1$, or $c-b=1$, which results in
$b=0$ and $c=1$.
Algebraically, $a=1$ is valid. However, geometrically there is no triangle with a side length equal to 0 .

| $\boldsymbol{a}$ | $\boldsymbol{b}$ |  | $\boldsymbol{c}$ | $\boldsymbol{a}^{\mathbf{2}}$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 0 |  | 1 | 1 |
| 3 | 4 |  | 5 | 9 |
| 5 | 12 |  | 13 | 25 |

Table 2: $a$ is even.
Before participants start Table 2, instruct them to leave the first row blank, and start the second row with $a$ is 4 . Work with your group to complete the three rows where $a=4,6$, and 8 . Be prepared to share in whole class discussion.

| $\boldsymbol{a}$ | $\boldsymbol{b}$ |  | $\boldsymbol{c}$ | $\boldsymbol{a}^{\mathbf{2}}+\boldsymbol{b}^{\mathbf{2}}$ | $=\boldsymbol{c}^{\mathbf{2}}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |  |  |
| 4 | 3 |  | 5 | 16 | 9 | 25 |
| 6 | 8 |  | 10 | 36 | 64 | 100 |
| 8 | 15 |  | 17 | 64 | 225 | 289 |

If needed, mention that the first row, $4,3,5$, is the only row in which $a$ is greater than $b$. Encourage participants to look for patterns in the same way as they did for Table 1.

When almost all participants have found the triples for $a=4,6,8$, conduct a whole group discussion about the patterns participants observed, again covering the $b^{2}$ and $c^{2}$ columns.

Have participants note that $b$ and $c$ differ by 2 , or, $c-b=2$, then $b+c=\frac{a^{2}}{2}$.
Usually, participants will divide $a^{2}$ by 2 , from the process used in Table 1, and then discover that they need to divide by 2 again to find a number to determine $b$ and $c$.
For example, if $a^{2}=36$, then divide 36 by 2 to get 18 . Divide 18 by 2 again to get 9 . Then add 1 to 9 to get $c=10$, and subtract 1 from 9 to get $b=8$.

Write 9 in the space between 8 and 10. Similarly, write 4 in the space between 3 and 5, and write 16 in the space between 15 and 17.

| $\boldsymbol{a}$ | $\boldsymbol{b}$ | Middle \# | $\boldsymbol{c}$ | $\boldsymbol{a}^{\mathbf{2}}$ |
| :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |
| 4 | 3 | 4 | 5 | 16 |
| 6 | 8 | 9 | 10 | 36 |
| 8 | 15 | 16 | 17 | 64 |

What operation performed on $a^{2}$ produces the number between $b$ and $c$ ? Divide by 4 .

What operation performed on the "middle number" produces $b$ and $c$ ?
Subtract 1 to get $b$; add 1 to get $c$.
Write the following numbers in the table. Participants find the missing values and then justify the answers.

| $\boldsymbol{a}$ | $\boldsymbol{b}$ | Middle \# | $\boldsymbol{c}$ | $\boldsymbol{a}^{\mathbf{2}}$ |
| :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |
| 4 | 3 | 4 | 5 | 16 |
| 6 | 8 | 9 | 10 | 36 |
| 8 | 15 | 16 | 17 | 64 |
| 10 |  |  |  |  |
|  | 35 |  |  |  |
|  |  |  | 50 |  |

For $a=10, a^{2}=100$.
Divide 100 by 4 . The "middle number" is 25 and $b=24, c=26$.
Check: $a^{2}+b^{2}=10^{2}+24^{2}$

$$
\begin{aligned}
& =100+576 \\
& =676 . \\
c^{2} & =26^{2}=676 . \\
a^{2}+b^{2} & =c^{2} .
\end{aligned}
$$

For $b=35, c=37$.
The "middle number" $=36$.
To get $a^{2}$, multiply by 4 .
$a^{2}=144$.
$a=12$.
Check: $a^{2}+b^{2}=12^{2}+35^{2}$

$$
\begin{aligned}
& =144+1225 \\
& =1369 . \\
c^{2} & =13^{2}=1369 . \\
a^{2}+b^{2} & =c^{2} .
\end{aligned}
$$

For $c=50, b=c-2$.
$b=48$.
The "middle number" $=49$.
To get $a^{2}$, multiply by 4 .
$a^{2}=196$.
$a=14$.
Check: $a^{2}+b^{2}=14^{2}+48^{2}$

| $a$ | $b$ | Middle \# | c | $a^{2}$ |
| :---: | :---: | :---: | :---: | :---: |
| 4 | 3 | 4 | 5 | 16 |
| 6 | 8 | 9 | 10 | 36 |
| 8 | 15 | 16 | 17 | 64 |
| 10 | 24 | 25 | 26 | 100 |
| 12 | 35 | 36 | 37 | 144 |
| 14 | 48 | 49 | 50 | 196 |

Participants may point out that the "middle number" is a sequence of perfect squares equal to $\left(\frac{a}{2}\right)^{2}$, which simplifies to $\frac{a^{2}}{4}$.

What is the triple in terms of $a$ ?

| $\boldsymbol{a}$ | $\boldsymbol{b}$ | middle \# | $\boldsymbol{c}$ | $\boldsymbol{a}^{\mathbf{2}}$ |
| :---: | :---: | :---: | :---: | :---: |
| $a$ | $\frac{a^{2}}{4}-1$ | $\frac{a^{2}}{4}$ | $\frac{a^{2}}{4}+1$ | $a^{2}$ |

Confirm the identity $a^{2}+\left(\frac{a^{2}}{4}-1\right)^{2}=\left(\frac{a^{2}}{4}+1\right)^{2}$.

$$
a^{2}+\left(\frac{a^{2}}{4}-1\right)^{2}
$$

$$
=a^{2}+\left(\frac{a^{4}}{16}-\frac{a^{2}}{2}+1\right)
$$

$$
=\frac{a^{4}}{16}+\frac{a^{2}}{2}+1
$$

$$
=\left(\frac{a^{2}}{4}+1\right)^{2}
$$

The table begins with $a=4$. The even whole numbers should begin with 2 .
Write the value $a=2$ in the first row of the table. If participants did not leave the first row blank, they can enter the $a=2$ row above the table headings.

| $\boldsymbol{a}$ | $\boldsymbol{b}$ | middle \# | $\boldsymbol{c}$ | $\boldsymbol{a}^{\mathbf{2}}$ |
| :---: | :---: | :---: | :---: | :---: |
| 2 |  |  |  |  |
| 4 | 3 | 4 | 5 | 16 |

What is the triple for $\boldsymbol{a}=\mathbf{2}$, the first even whole number?
If $a=2$, then $a^{2}=4, b=0$, and $c=2$.
Algebraically 2, 0,2 is a valid triple, but geometrically 0 is not a valid length measurement for the side of a triangle.

| $\boldsymbol{a}$ | $\boldsymbol{b}$ | middle \# | $\boldsymbol{c}$ | $\boldsymbol{a}^{\mathbf{2}}$ |
| :---: | :---: | :---: | :---: | :---: |
| 2 | 0 | 1 | 2 | 4 |
| 4 | 3 | 4 | 5 | 16 |

Point out that in Table 1 ( $a$ is odd), the triples are all primitive. This means that the three numbers have no common factors. However, in the second table ( $a$ is even), every other row is primitive, while those in between are all multiples of 2 . When 2 is factored out, the other factor is one of the triples from Table 1.

|  | $\boldsymbol{a}$ | $\boldsymbol{b}$ | $\boldsymbol{c}$ |
| :---: | :---: | :---: | :---: |
|  |  |  |  |
| Primitive | 4 | 3 | 5 |
| $2[3,4,5]$ | 6 | 8 | 10 |
| Primitive | 8 | 15 | 17 |
| $2[5,12,13]$ | 10 | 24 | 26 |
| Primitive | 12 | 35 | 37 |
| $2[7,24,25]$ | 14 | 48 | 50 |

## Explain why multiples of triples always satisfy the Pythagorean Theorem?

In symbolic terms, explain why $a^{2}+b^{2}=c^{2}$, is $(k a)^{2}+(k b)^{2}=(k c)^{2}$ is always true for any value of $\boldsymbol{k}$.
$(k a)^{2}+(k b)^{2}$
$=k^{2} a^{2}+k^{2} b^{2}$
$=k^{2}\left(a^{2}+b^{2}\right)$
$=k^{2}\left(c^{2}\right)$
$=k^{2} c^{2}$
$=(k c)^{2}$.
The patterns for " $a$ is odd" and " $a$ is even" represent a few of the infinite numbers of Pythagorean triples. These also represent a large number of triples that typically appear in high school texts or test problems. Another common triple is 20, 21, 29, which does not fit either of the patterns.

Table 3 is provided for participants for optional further investigation and notes.
Set up the triple, 20, 21, 29, in a comparable table.

| $\boldsymbol{a}$ | $\boldsymbol{b}$ | Middle \# | $\boldsymbol{c}$ | $\boldsymbol{a}^{\mathbf{2}}$ |
| :---: | :---: | :---: | :---: | :---: |
| 20 | 21 | 25 | 29 | 400 |

Consider the relationships from Tables 1 and 2. What are the comparable relationships here?
In Table $1, c-b=1$, and the "middle number" is $\frac{a^{2}}{2}$.
In Table 2, $c-b=2$, and the "middle number" is $\frac{a^{2}}{4}$.
In this problem, $c-b=8$, and the "middle number" is $\frac{a^{2}}{16}$.

Since the "middle number" is a perfect square, find other triples that fit the 20, 21, 29 pattern.

| $a$ | $b$ | Middle \# | $\boldsymbol{c}$ | $\boldsymbol{a}^{2}$ |
| :---: | :---: | :---: | :---: | :---: |
| 20 | 21 | 25 | 29 | 400 |
|  |  | 4 |  |  |
|  |  | 9 |  |  |
|  |  | 16 |  |  |
|  |  | 36 |  |  |
|  |  | 49 |  |  |


| $\boldsymbol{a}$ | $\boldsymbol{b}$ | middle \# | $\boldsymbol{c}$ | $\boldsymbol{a}^{\mathbf{2}}$ |  |
| :---: | :---: | :---: | :---: | :---: | :--- |
| $\mathbf{2 0}$ | $\mathbf{2 1}$ | $\mathbf{2 5}$ | $\mathbf{2 9}$ | $\mathbf{4 0 0}$ | Primitive |
| 8 | 0 | $\mathbf{4}$ | 8 | 64 | Not a triangle |
| 12 | 5 | $\mathbf{9}$ | 13 | 144 | $5,12,13$ in Table 1 |
| 16 | 12 | $\mathbf{1 6}$ | 20 | 256 | $3[4,3,5]$ |
| 24 | 32 | $\mathbf{3 6}$ | 40 | 576 | $8[3,4,5]$ |
| 28 | 45 | $\mathbf{4 9}$ | 53 | 784 | Primitive |

According to Pappas (1991, p. 79), the ancient Greeks discovered:
If $m$ is an odd natural number, then $\left(\frac{m^{2}+1}{2}\right)^{2}=\left(\frac{m^{2}-1}{2}\right)^{2}+m^{2}$.
The patterns in Table 1 satisfy this identity.
Plato's formula, where $m$ is a natural number is:
$\left(m^{2}+1\right)^{2}=\left(m^{2}-1\right)^{2}+(2 m)^{2}$.
The patterns in Table 2, where $a=2 m$, an even number, satisfy this identity.
Euclid's method:
If $x$ and $y$ are integers and if $a=x^{2}-y^{2}, b=2 x y, c=x^{2}+y^{2}$, then $a, b$, and $c$, are integers, such that $a^{2}+b^{2}=c^{2}$.
You may interchange $a$ and $b$ in the formula, so that $a$ can be odd for some cases of $a=x^{2}-y^{2}$ or $a$ can be even $(a=2 x y)$.

Further explorations may be conducted by making generalizations for the $c-b$ value and the corresponding divisibility of $a^{2}$.

Success in this activity indicates that participants are working at the Relational Level because they use the Pythagorean Theorem to verify the triples. Since algebraic deductive reasoning is used to generalize the triples, they may also be working at the Deductive Level.

## Pythagorean Triples

When three whole numbers satisfy the Pythagorean Theorem, they are called Pythagorean triples or triplets. Plato and Euclid derived formulas for the triples. At the end of this activity, you will examine these formulas.
Interesting patterns and generalizations exist in certain groups of triples. Let $a$ and $b$ represent the lengths of the legs of a right triangle, and let $c$ represent the length of the hypotenuse. The two tables below are grouped according to the value of $a$. In the first table $a$ is odd; in the second table $a$ is even.

Table 1: $a$ is odd.
Work with your group to complete the three rows where $a=3,5$ and 7 . Be prepared to share in whole class discussion.

| $\boldsymbol{a}$ | $\boldsymbol{b}$ |  | $\boldsymbol{c}$ | $\boldsymbol{a}^{\mathbf{2}}+\boldsymbol{b}^{\mathbf{2}}$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  | $\boldsymbol{c}^{\mathbf{2}}$ |  |
| 3 | 4 |  |  |  |  |  |
| 5 | 12 |  |  |  |  |  |
| 7 |  |  | 25 |  |  |  |
|  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |

Notes:

Table 2: $a$ is even.
Work with your group to complete the three rows where $a=4,6$, and 8 . Be prepared to share in whole class discussion.

| $\boldsymbol{a}$ | $\boldsymbol{b}$ | middle \# | $\boldsymbol{c}$ | $\boldsymbol{a}^{\mathbf{2}}+\boldsymbol{b}^{\mathbf{2}}$ |  | $=\boldsymbol{c}^{\mathbf{2}}$ |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |

Notes:

Table 3:


Notes:

## Special Right Triangles

Overview: In this activity participants investigate special right triangles.

## Objective: TExES Mathematics Competencies

II.006.G. The beginning teacher models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology.
III.011.D. The beginning teacher applies the Pythagorean theorem, proportional reasoning, and right triangle trigonometry to solve measurement problems.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

## Geometry TEKS

b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.C. The student demonstrates what it means to prove mathematically that statements are true.
c.3. The student identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-6090) and triangles whose sides are Pythagorean triples. e.1.C. The student develops, extends, and uses the Pythagorean Theorem.
f.3. In a variety of ways, the student develops, applies, and justifies triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples.

Background: Participants need to have knowledge of the Pythagorean Theorem.
Materials: geoboard or geoboard dot paper (provided in the Appendix), unlined 8.5 in. by 11 in. paper

## New Terms:

## Procedures:

This activity is taken from Discovering Geometry: An Investigative Approach, $3^{\text {rd }}$ Edition, © 2003, p. 475 with permission from Key Curriculum Press.

## Part 1

On your paper, sketch an isosceles right triangle. Label the legs $l$ and the hypotenuse $h$.


1. Pick any positive integer for $l$, the length of the legs of an isosceles right triangle.

Show this triangle on a geoboard or on geoboard dot paper. Use the Pythagorean Theorem to find $h$. Simplify the square root.

$$
\begin{aligned}
l^{2}+l^{2} & =h^{2} \\
2^{2}+2^{2} & =h^{2} \\
4+4 & =h^{2} \\
8 & =h^{2} \\
\sqrt{8} & =h \\
2 \sqrt{2} & =h
\end{aligned}
$$


2. Repeat 1 with several other values for $l$. Share results with your group. Do you see a pattern in the relationship between $l$ and $h$ ?

$$
\begin{aligned}
l^{2}+l^{2} & =h^{2} \\
3^{2}+3^{2} & =h^{2} \\
9+9 & =h^{2} \\
18 & =h^{2} \\
\sqrt{18} & =h \\
3 \sqrt{2} & =h
\end{aligned}
$$


3. State your conclusion in words and using a drawing.

In a $45^{\circ}-45^{\circ}-90^{\circ}$ right triangle, the length of the hypotenuse is $\sqrt{2}$ times the length of the leg.

## Part 2

On your paper, sketch an equilateral triangle, $\triangle A B C$. Draw the altitude $\overline{B D}$ from vertex $B$ to side $\overline{A C}$. On your sketch, mark congruent sides and show the measures of all angles in $\triangle A B D$ and $\triangle C B D$.

4. What is the relationship between $\overline{A B}$ and $\overline{A D}$ ? Will this relationship always exist in a $30^{\circ}-60^{\circ}-90^{\circ}$ triangle? Explain.
Since $A C=A B$, then $A D=\frac{1}{2} A B$.
5. Sketch a $30^{\circ}-60^{\circ}-90^{\circ}$ triangle. Choose any positive integer for the length of the shorter leg. Use the relationship from 4 together with the Pythagorean Theorem to find the length of the other leg. Simplify the square root.

$$
\begin{aligned}
6^{2}+l^{2} & =12^{2} \\
36+l^{2} & =144 \\
l^{2} & =108 \\
l & =6 \sqrt{3}
\end{aligned}
$$


6. Repeat 5 with several values for the length of the shorter leg. Share results with the group.

$$
\begin{aligned}
9^{2}+l^{2} & =18^{2} \\
81+l^{2} & =324 \\
l^{2} & =243 \\
l & =9 \sqrt{3}
\end{aligned}
$$


7. Write a generalization for relating the sides of any $30^{\circ}-60^{\circ}-90^{\circ}$ triangle.

In a $30^{\circ}-60^{\circ}-90^{\circ}$ right triangle, the length of the hypotenuse is twice the length of the short leg. The length of the long leg is $\sqrt{3}$ times the length of the short leg.
8. Use algebra to verify the relationship for any $30^{\circ}-60^{\circ}-90^{\circ}$ triangle by using the triangle to the right.

$$
\begin{aligned}
(2 a)^{2} & =a^{2}+b^{2} \\
4 a^{2} & =a^{2}+b^{2} \\
3 a^{2} & =b^{2} \\
a \sqrt{3} & =b
\end{aligned}
$$



Success in this activity indicates that participants are working at the Relational Level because they use both inductive and deductive methods to generalize properties for special right triangles.

## Special Right Triangles

## Part 1

On your paper, sketch an isosceles right triangle. Label the legs $l$ and the hypotenuse $h$.

1. Pick any positive integer for $l$, the length of the legs of an isosceles right triangle. Show this triangle on a geoboard or on geoboard dot paper. Use the Pythagorean Theorem to find $h$. Simplify the square root.
2. Repeat 1 with several other values for $l$. Share results with your group. Do you see a pattern in the relationship between $l$ and $h$ ?
3. State your conclusion in words and using a drawing.

## Part 2

On your paper, sketch an equilateral triangle, $\triangle A B C$. Draw the altitude $\overline{B D}$ from vertex $B$ to side $\overline{A C}$. On your sketch, mark congruent sides and show the measures of all angles in $\triangle A B D$ and $\triangle C B D$.
4. What is the relationship between $\overline{A B}$ and $\overline{A D}$ ? Will this relationship always exist in a $30^{\circ}-60^{\circ}-90^{\circ}$ triangle? Explain.
5. Sketch a $30^{\circ}-60^{\circ}-90^{\circ}$ triangle. Choose any positive integer for the length of the shorter leg. Use the relationship from 4 together with the Pythagorean Theorem to find the length of the other leg. Simplify the square root.
6. Repeat 5 with several values for the length of the shorter leg. Share results with the group.
7. Write a generalization relating the sides of any $30^{\circ}-60^{\circ}-90^{\circ}$ triangle.
8. Use algebra to verify the relationship for any $30^{\circ}-60^{\circ}-90^{\circ}$ triangle by using the triangle to the right.


## Distance Formula

Overview: In this activity participants investigate the length of line segments using the Pythagorean Theorem.

## Objective: TExES Mathematics Competencies

III.011.D. The beginning teacher applies the Pythagorean theorem, proportional reasoning, and right triangle trigonometry to solve measurement problems.
III.014.E. The beginning teacher applies concepts and properties of slope, midpoint, parallelism, perpendicularity, and distance to explore properties of geometric figures and solve problems in the coordinate plane.

## Geometry TEKS

d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.
d.2.C. The student develops and uses formulas including distance and midpoint.
e.1.C. The student develops, extends, and uses the Pythagorean Theorem.

Background: Participants need to be able to understand and use the Pythagorean Theorem.

Materials: centimeter grid paper, centimeter grid transparency, 3 in. square adhesive notes in two colors (one of each color per participant)

New Terms: distance formula

## Procedures:

Participants find the length of a non-vertical and non-horizontal line segment whose endpoints lie on a given coordinate plane. A right triangle is drawn using the original line segment as the hypotenuse. Horizontal and vertical segments drawn along the gridlines from the endpoints of the original segment form the legs. The lengths of the legs can be determined by observation. The length of the original segment can be found using the Pythagorean Theorem.

When the segment has endpoints which do not fit on a given coordinate plane, participants develop and apply the distance formula, based on an understanding of the Pythagorean Theorem.

Prior to the training, write an ordered pair on an adhesive note for each participant. On one color adhesive note write an ordered pair whose $x$ - and $y$-values vary from -8 to 8 . On the other color adhesive note, write an ordered pair whose $x$ - and $y$-values lie between 10
and 50 or between -10 and -50 . Try to provide a wide variety of ordered pair choices for the whole class.

## Part 1

Begin the activity by distributing an adhesive note in each color to each participant. Each group of four participants will then have four ordered pairs. On centimeter grid paper, using the coordinates between -8 and 8 , ask participants to create four line segments using the ordered pairs as endpoints. It is possible to create six segments. Each participant will find the distances between the endpoints.

Lead a whole class discussion in which participants summarize how they found the distance. Possible mathematical relationships to highlight are:

- For non-vertical or non-horizontal lines, draw a vertical line segment and horizontal line segment along the grid lines to form a right triangle. Use the Pythagorean Theorem to find the length of the hypotenuse.
- To find the lengths of the vertical and horizontal legs, count the grid units from endpoint to endpoint.
- Irrational numbers should be simplified, e.g., $\sqrt{24}$ is written $2 \sqrt{6}$.


## Part 2

Ask participants to repeat the activity using the ordered pairs on the other color adhesive note, having coordinate values greater than 50 or less than -50 . Give participants time to grapple with the fact that the grid paper is too small to plot the points unless the scale is changed.

Ask participants to write an expression for the length of the horizontal leg using the $x$ coordinates. Then write a similar expression for the length of the vertical leg using the $y$ coordinates. Then apply the leg measurements to the Pythagorean Theorem to find the distance between the points. Use the expressions to write a general formula for distance.

Lead a whole class discussion in which participants summarize how the distance was found. Ask a participant to describe how his/her group developed the expressions, using the group's examples from Part 2.

Given the coordinates of the endpoints of a line segment, $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$, the length of the line segment is given by $\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}}$. This is commonly known as the distance formula. Remind participants to add the term distance formula to their glossaries.

Success with this activity indicates that participants are performing at the Relational Level since they interrelate algebraic and geometric representations.

## Applying Pythagoras, Part II

Overview: Participants apply the Pythagorean Theorem at various levels of challenge.

## Objective: TExES Mathematics Competencies

III.011.B. The beginning teacher applies formulas for perimeter, area, surface area, and volume of geometric figures and shapes (e.g., polygons, pyramids, prisms, cylinders, cones, spheres) to solve problems.
III.011.D. The beginning teacher applies the Pythagorean theorem, proportional reasoning, and right triangle trigonometry to solve measurement problems.
III.013.D. The beginning teacher computes the perimeter, area, and volume of figures and shapes created by subdividing and combining other figures and shapes (e.g., arc length, area of sectors).
III.018.E. The beginning teacher understands the problem-solving process (i.e., recognizing that a mathematical problem can be solved in a variety of ways, selecting an appropriate strategy, evaluating the reasonableness of a solution).
III.019.E. The beginning teacher understands the use of visual media, such as graphs, tables, diagrams, and animations, to communicate mathematical information.

## Geometry TEKS

b.4. The student selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems. c.1. The student identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-6090) and triangles whose sides are Pythagorean triples.
d.2.C. The student develops and uses formulas including distance and midpoint.
e.1.A. The student finds areas of regular polygons and composite figures.
e.1.C. The student develops, extends, and uses the Pythagorean Theorem.
f.3. In a variety of ways, the student develops, applies, and justifies triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples.

Background: Participants need the ability to apply the Pythagorean Theorem in problems that integrate properties of other planar figures such as triangles, quadrilaterals, and to some extent circles.

Materials: calculator

## New Terms:

## Procedures:

Participants work independently or in pairs. The problems may be assigned for homework.

1. What is the area of trapezoid $P Q R S$, whose measures are shown in the diagram?


Draw the perpendicular segment from $Q$ or $R$ to $\overline{P S}$. The height of the trapezoid is the same as the height of the $30^{\circ}-60^{\circ}-90^{\circ}$ triangle: $4 \sqrt{3}$ units. The area of trapezoid PQRS is $84 \sqrt{3}$ square units.
2. The home plate used in baseball can be produced by adding two isosceles right triangles to a square as shown. What is the area of home plate in square feet?

The hypotenuses of the isosceles right

triangles measure 1 ft . Using the 1:1: $\sqrt{2}$ ratio, the legs of the triangles measure $\frac{\sqrt{2}}{2}$
ft. Therefore the area of each triangle is $\frac{1}{2} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}=\frac{1}{4} f t^{2}$. The area of the home plate is the sum of the areas of the 1 ft by 1 ft square and the two isosceles right triangles. Therefore, the area is $1.5 \mathrm{ft}^{2}$.
3. An equilateral triangle has a perimeter of 24 cm . What is the area in square centimeters of the triangle formed by connecting the midpoints of the sides of the original triangle? Express your answer in simplest radical form.


Draw an equilateral triangle and the triangle formed by connecting the midpoints of its sides. Draw the altitude. Use the 1: $\sqrt{3}$ $: 2$ ratio for a $30^{\circ}-60^{\circ}-90^{\circ}$ triangle to find the length of the altitude, $2 \sqrt{3} \mathrm{~cm}$. The area of the small triangle is $\frac{1}{2} \cdot 2 \sqrt{3} \cdot 4=4 \sqrt{3} \mathrm{~cm}^{2}$.
4. A 25 -foot support beam leans against a wall as shown. The base of the beam is 7 feet from the wall. If the top of the beam is lowered 4 feet, how many feet farther away from the wall will the base of the beam be after it is lowered?


As shown in the diagram, the beam forms a 7:24:25 triangle. When it is lowered the beam forms a 15:20:25 triangle. The base of the beam has moved 8 feet farther away from the wall.
5. The diagram consists of three nested squares. Find the ratio of the area of the smallest square to the area of the largest square.

The side length of the middle square is equal
 to the length of the hypotenuse of the right triangle with legs of length 3 and 4 units. The side length of the middle square is 5 units.
The side length of the smallest square is equal to the length of the hypotenuse of a right triangle with legs of length 2 and 3 units, which is $\sqrt{13}$.
The ratio of the area of the smallest square to the area of the largest square is $\frac{13}{49}$.
6. Lightning hit a tree one-fourth of the distance up the trunk from the ground, breaking the tree so that its top landed at a point 60 feet from its base, as shown. How many feet tall was the tree originally?

Let the height of the upright portion of the tree be $x$, then the hypotenuse is $3 x$. Apply the Pythagorean Theorem, using leg lengths $x$ and 60, and hypotenuse length $3 x$.
$x^{2}+60^{2}=(3 x)^{2}$. Solve for $x$. The original height of the tree was $4 x$, approximately 84.8 feet.
7. Find the perimeter of the polygon shown. The triangles are all right triangles.

See the activity sheet for a clear picture of the figure.
The figure is comprised of a series of $30^{\circ}-60^{\circ}-90^{\circ}$ triangles. The hypotenuse of the largest triangle measures 64 units. The long leg of the second largest triangle measures $16 \sqrt{3}$ units. The long legs of the third, fourth, fifth and sixth triangles measure $8 \sqrt{3}, 4 \sqrt{3}, 2 \sqrt{3}$, and $\sqrt{3}$ units. The perimeter is $63+63 \sqrt{3}$, the sum of the lengths of the long legs of the triangles and the 63 units remaining on the hypotenuse of the largest triangle.
8. Equilateral triangle $Q R S$ is inscribed in circle $O$. The radius of the circle is 8 units. The area of the shaded region can be expressed as $x \pi-y \sqrt{3}$, where $x$ and $y$ are positive integers. Find the ordered pair $(x, y)$.


Draw an altitude of the triangle. The center of the circle $O$ divides the altitude into lengths that are in ratio of $2: 1$. The length of the radius is 8 units. The length of the altitude is 12 units and the length of a side of the triangle is $8 \sqrt{3}$ units.
The area of the shaded region is
$8^{2} \cdot \pi-\frac{1}{2} \cdot 12 \cdot 8 \sqrt{3}$ or $64 \pi-48 \sqrt{3}$ square units.
The ordered pair is $(64,48)$.
9. A circular table is pushed into the corner of a square room so that a point $P$ on the edge of the table is 8 inches from one wall and 9 inches from the other wall as shown. Find the radius of the circular table in inches.


Draw the radius to $P$, and the horizontal and vertical legs of the right triangle. The radius is the hypotenuse. The legs are shorter than the radius by 8 and 9 inches respectively. Use the Pythagorean Theorem, $(r-9)^{2}+(r-8)^{2}=r^{2}$.
Simplifying: $r^{2}-34 r+145=0$ or $(r-29)(r-5)=0$. This equation has two solutions $r=29$ and $r=5$. Only $r=29$ makes sense in the context of the problem so the radius of the table has a length of 29 inches.
10. A square with $6^{\prime \prime}$ sides is shown. If $P$ is a point in the interior of the square such that the segments $\overline{P A}, \overline{P B}$, and $\overline{P C}$ are equal in length, and $\overline{P C}$ is perpendicular to $\overline{F D}$, what is the area, in square inches, of $\triangle A P B$ ?


Let $C P=x$.
Extend $\overline{C P}$ to intersect $\overline{A B}$ at $Q$.
Altitude $P Q=6-x$.
$A P=x . A Q=3$.
Apply the Pythagorean Theorem to the measures of $\triangle A P Q$ to solve for $x$.
$x=3 \frac{3}{4}$ or $\frac{15}{4}$ inches.
Therefore the area of $\triangle A P B=$

$$
\frac{1}{2} \cdot 3 \cdot\left(6-\frac{15}{4}\right)=\frac{1}{2} \cdot 3 \cdot \frac{9}{4}=\frac{27}{8} \mathrm{in}^{2}
$$

11. The square shown is externally tangent to the smaller circle and internally tangent to the larger circle. Find the ratio of the area of the smaller circle to the area of the larger circle.


Draw the radius, $r$, of the small circle to the midpoint of a side of the square. Draw the radius of the large circle to a vertex of the square, forming a $45^{\circ}-45^{\circ}-90^{\circ}$ right triangle, therefore this radius has a measure of $r \sqrt{2}$.
The ratio of the areas of the smaller circle to the larger circle is $\frac{\pi \cdot r^{2}}{\pi(r \sqrt{2})^{2}} \frac{\pi r^{2}}{\pi 2 r^{2}}=\frac{1}{2}$.
12. A rectangular solid has dimensions 3 cm by 4 cm by 12 cm . What is the length, in centimeters, of the diagonal $\overline{A B}$ of the solid?


Draw a face diagonal along one of the 3 cm by 4 cm faces. The face diagonal measures 5 cm .
The face diagonal, the 12 cm edge, and $\overline{A B}$ are
the sides of a right triangle. $A B=13 \mathrm{~cm}$.
13. Point $A$ is located in 3-dimensional space at coordinate position (4, 4, 4). Find the distance from point $A$ to the origin.


The distance from $A$ to the origin is $\sqrt{4^{2}+4^{2}+4^{2}}=4 \sqrt{3}$ units.
14. The semicircle shown has center $O$ and diameter $\overline{A B}$. A rectangle whose length is twice its width has two vertices on the semicircle. Find the ratio of the area of the semicircle to the area of the rectangle.


Draw the radius to a vertex of the rectangle on the semicircle. Let the width of the rectangle be $x$ units. The ratio of the area of the semicircle to the area of the rectangle is $\frac{\pi x^{2}}{2 x^{2}}=\frac{\pi}{2}$.

Success with this activity indicated that participants are working at the van Hiele Relational Level, because they use properties and formulas of various figures to solve the problems.

## Applying Pythagoras, Part II

1. What is the area of trapezoid $P Q R S$, whose measures are shown in the diagram?

2. The home plate used in baseball can be produced by adding two isosceles right triangles to a square as shown. What is the area of home plate in square feet?

3. An equilateral triangle has a perimeter of 24 cm . What is the area in square centimeters of the triangle formed by connecting the midpoints of the sides of the original triangle? Express your answer in simplest radical form.
4. A 25 -foot support beam leans against a wall as shown. The base of the beam is 7 feet from the wall. If the top of the beam is lowered 4 feet, how many feet farther away from the wall will the base of the beam be after it is lowered?

5. The diagram consists of three nested squares. Find the ratio of the area of the smallest square to the area of the largest square.

6. Lightning hit a tree one-fourth of the distance up the trunk from the ground, breaking the tree so that its top landed at a point 60 feet from its base, as shown. How many feet tall was the tree originally?

7. Find the perimeter of the polygon shown. The triangles are all right triangles.

8. Equilateral triangle $Q R S$ is inscribed in circle $O$. The radius of the circle is 8 units. The area of the shaded region can be expressed as $x \pi-y \sqrt{3}$, where $x$ and $y$ are positive integers. Find the ordered pair $(x, y)$.

9. A circular table is pushed into the corner of a square room so that a point $P$ on the edge of the table is 8 inches from one wall and 9 inches from the other wall as shown. Find the radius of the circular table in inches.

10. A square with $6 "$ sides is shown. If $P$ is a point in the interior of the square such that the segments $\overline{P A}, \overline{P B}$, and $\overline{P C}$ are equal in length, and $\overline{P C}$ is perpendicular to $\overline{F D}$, what is the area, in square inches, of $\triangle A P B$ ?

11. The square shown is externally tangent to the smaller circle and internally tangent to the larger circle. Find the ratio of the area of the smaller circle to the area of the larger circle.

12. A rectangular solid has dimensions 3 cm by 4 cm by 12 cm . What is the length, in centimeters, of the diagonal $\overline{A B}$ of the solid?

13. Point $A$ is located in 3-dimensional space at coordinate position $(4,4,4)$. Find the distance from point $A$ to the origin.
14. The semicircle shown has center $O$ and diameter $\overline{A B}$. A rectangle whose length is twice its width has two vertices on the semicircle. Find the ratio of the area of the semicircle to the area of the rectangle.


## References and Additional Resources

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## Unit 7 - Polygons and Circles

## Diagonals of a Polygon

Overview: In this activity, participants investigate the number of diagonals from a given vertex of a polygon.

## Objective: TExES Mathematics Competencies

II.004.A. The beginning teacher recognizes and extends patterns and relationships in data presented in tables, sequences, or graphs. II.006.G. The beginning teacher models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology.
III.011.A. The beginning teacher applies dimensional analysis to derive units and formulas in a variety of situations (e.g., rates of change of one variable with respect to another) and to find and evaluate solutions to problems.
III.013.A. The beginning teacher analyzes the properties of polygons and their components.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
IV.015.B. The beginning teacher organizes, displays, and interprets data in a variety of formats (e.g., tables, frequency distributions, scatter plots, stem-and-leaf plots, box-and-whisker plots, histograms, pie charts).
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.
V.019.D. The beginning teacher communicates mathematical ideas using a variety of representations (e.g., numeric, verbal, graphical, pictorial, symbolic, concrete).

## Geometry TEKS

b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.D. The student uses inductive reasoning to formulate a conjecture.
b.4. The student uses a variety of representations to describe geometric relationships and solve problems.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
d.2.C. The student develops and uses formulas including distance and midpoint.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.

Background: Participants should know that the sum of the measures of the angles in any given triangle is $180^{\circ}$. They should also be familiar with terms such as vertex, diagonal, and interior angle.

Materials: straightedge, graphing calculator

## New Terms:

## Procedures:

In this activity, participants will investigate the relationship between the number of sides of a polygon and the total number of diagonals.

1. Complete the following table relating the numbers of sides of a polygon to the total number of diagonals.

| Number of sides of <br> the polygon | $\mathbf{3}$ | $\mathbf{4}$ | $\mathbf{5}$ | $\mathbf{6}$ | $\mathbf{7}$ | $\mathbf{8}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Number of <br> diagonals in the <br> polygon | 0 | 2 | 5 | 9 | 14 | 20 |

Participants may need to create the polygons and physically draw the diagonals to be able to complete the first few columns of the chart.

Create a scatter plot of your data on your graphing calculator. Place the number of sides of the polygon on the $x$-axis and the number of diagonals on the $y$-axis.

For an intensive tutorial on the use of the graphing calculator see Utilizing the graphing calculator in the secondary mathematics and science classroom at http://www.esc4.net/math.

Using the given window the scatter plot should look as follows:


What patterns do you observe?
Graphically and numerically participants should see that the data is not linear. Finite differences (shown above) illustrate that the first finite differences are not constant; however the second finite differences have a constant value of one. This indicates that the data is quadratic, a fact that appears to be confirmed by the scatter plot.
2. How many diagonals are in a 24 -gon? How many in an $n$-gon?

A 24-gon will have 252 diagonals. An n-gon will have $\frac{n(n-3)}{2}$ diagonals.
You may need to tell participants that a 24 -gon is a polygon with 24 sides and an $n$-gon is a polygon with $n$-sides. Although participants may want to extend the pattern to find the number of diagonals in a 24 -gon, they will soon realize that this is too tedious and that finding a rule to determine the number of diagonals would be more efficient. Participants may mistakenly believe from the scatter plot that the equation is a quadratic function with vertex at $(3,0)$. Others may attempt to perform a quadratic regression using the calculator to determine the function. If necessary, encourage participants to think about the underlying geometry in the situation.

## How many diagonals can be drawn from each vertex in an n-gon?

$n-3$ diagonals can be drawn from each vertex.
How many vertices are there in an $\boldsymbol{n}$-gon?
An $n$-gon has $n$ vertices.
Is the total number of diagonals $\boldsymbol{n}(\boldsymbol{n}-3)$ ?
No. Multiplying the number of vertices by the number of diagonals that can be drawn from each vertex "double counts" each diagonal. Therefore the total number of diagonals is $\frac{n(n-3)}{2}$.

Participants may have other approaches for determining this function. For example, some may see this as a combination problem. Determining the total number of segments that can be drawn among $n$ vertices is equivalent to determining the number of groups of 2 that can be taken from a group of $n$. This is equivalent to $\frac{n(n-1)}{2}$. The number of segments that are not diagonals, the $n$ sides, results in an
expression of $\frac{n(n-1)}{2}-n$. If a participant uses this approach, demonstrate that this expression is algebraically equivalent to $\frac{n(n-3)}{2}$.
3. Use your graphing calculator to verify your function for determining the number of diagonals in an $n$-gon.


The graphing calculator may be used in several ways to verify the function. We used the variable n in this activity, because we were dealing with discrete data. To use the calculator, we will need to use the variable $x$ which is more often associated with continuous functions. Participants may use a calculator table to see that their function reproduces their original table. They may also overlay their previously created scatter plot upon the graph of the function. The continuous function $f(x)=\frac{x(x-3)}{2}$ contains all the points of the discrete function for our geometric situation.
4. How many sides does a polygon with 860 diagonals have?

The polygon has 43 sides.
Participants have a variety of methods for finding this solution. They may use the table of values generated by the calculator to determine the number of sides. Or they may find the intersection point of the function $f(x)=\frac{x(x-3)}{2}$ and the line $g(x)=860$. Or they may solve the quadratic equation $. \frac{x(x-3)}{2}=860$.

In this activity participants are working at the van Hiele Descriptive Level as they determine properties of polygons. When they connect this with algebra, they approach the Relational Level.

## Diagonals of a Polygon

1. Complete the following table relating the numbers of sides of a polygon to the total number of diagonals.

| Number of <br> sides of the <br> polygon | 3 | 4 | 5 | 6 | 7 | 8 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Number of <br> diagonals in <br> the polygon |  |  |  |  |  |  |

Create a scatter plot of your data on your graphing calculator. Place the number of sides on the polygon on the $x$-axis and the number of diagonals on the $y$-axis.

What patterns do you observe?
2. How many diagonals are in a 24 -gon? How many in an $n$-gon?
3. Use your graphing calculator to verify your function for determining the number of diagonals in an $n$-gon.
4. How many sides does a polygon with 860 diagonals have?

Overview: In this activity, participants investigate the sum of the measures of the interior and exterior angles of a polygon.

## Objective: TExES Mathematics Competencies <br> II.004.A. The beginning teacher recognizes and extends patterns and relationships in data presented in tables, sequences, or graphs. <br> II.006.G. The beginning teacher models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology. <br> II.007.A. The beginning teacher recognizes and translates among various representations (e.g., written, tabular, graphical, algebraic) of polynomial, rational, radical, absolute value, and piecewise functions. III.013.A. The beginning teacher analyzes the properties of polygons and their components. <br> III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles). <br> V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas. <br> V.019.D. The beginning teacher communicates mathematical ideas using a variety of representations (e.g., numeric, verbal, graphical, pictorial, symbolic, concrete).

## Geometry TEKS

b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.D. The student uses inductive reasoning to formulate a conjecture.
c.1.A. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
d.2.C. The student develops and uses formulas including distance and midpoint.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.

Background: Participants need to have knowledge of the sum of the measures of a triangle's interior angles.

Materials: graphing calculator, straightedge, unlined 8.5 in . by 11 in . paper, scissors, tape, transparencies "Constructing a Polygon's Exterior Angles" and "Determining the Sum of a Polygon's Exterior Angles"
New Terms:

## Procedures:

In this activity, participants will use information about the number of diagonals of a polygon from one vertex to determine the sum of the measures of the interior angles of a polygon as well as the measure of each interior angle of a regular polygon. Recall the definition of a regular polygon as a polygon with all sides congruent and all angles congruent.

Provide participants with unlined 8.5 in . by 11 in . paper.

1. In your group, draw polygons with 3 to 7 sides. Use a straightedge to draw all the diagonals from one vertex. Share your results to complete the table below.

| Number of <br> sides of the <br> polygon | $\mathbf{3}$ | $\mathbf{4}$ | $\mathbf{5}$ | $\mathbf{6}$ | $\mathbf{7}$ | $\ldots$ | $\boldsymbol{n}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Number of <br> diagonals <br> from one <br> vertex | 0 | 1 | 2 | 3 | 4 | $n-3$ |  |
| Number of <br> triangles <br> created | 1 | 2 | 3 | 4 | 5 | $n-2$ |  |
| Total <br> measure of <br> interior <br> angles <br> (degrees) | $1 \cdot 180$ | $2 \cdot 180$ | $3 \cdot 180$ | $4 \cdot 180$ | $5 \cdot 180$ | $(n-2) \cdot 180$ |  |

2. Using the formula you derived in 1 for the sum of the measures of the interior angles of a polygon, complete the table below to determine the measure of each interior angle of a regular polygon.

| Number <br> of sides of <br> the <br> polygon | $\mathbf{3}$ | $\mathbf{4}$ | $\mathbf{5}$ | $\mathbf{6}$ | $\mathbf{7}$ | $\boldsymbol{\cdots}$ | $\boldsymbol{n}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Sum of <br> measures <br> of <br> interior <br> angles <br> (degrees) | 180 | 360 | 540 | 720 | 900 | $180(n-2)$ |  |


| Measure <br> of an <br> interior <br> angle <br> (degrees) | $\frac{180}{3}=$ | $\frac{360}{4}=$ | $\frac{540}{5}=$ | $\frac{720}{6}=$ | $\frac{900}{7}=$ | $\frac{180(n-2)}{n}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |

Stress with participants that the formula for the sum of the measures of interior angles is true for all polygons, but that this formula is applicable only to regular polygons, since by definition, all of the angles in a regular polygon must be congruent.
3. If $n=50$, what is the measure of each interior angle of a regular polygon?

The measure of the interior angle of a 50-gon is $\frac{180^{\circ}(50-2)}{50}$ or $172.8^{\circ}$.
4. If $n=100$, what is the measure of each interior angle of a regular polygon?

The measure of the interior angle of a 100-gon is $\frac{180^{\circ}(100-2)}{100}$ or $176.4^{\circ}$.
5. Is it possible for a regular polygon to have an interior angle measure of $175.5^{\circ}$ ?

Explain.
$175.5^{\circ}=\frac{180^{\circ}(n-2)}{n}$
$175.5^{\circ} n=180^{\circ} n-360^{\circ}$
$360^{\circ}=4.5^{\circ} n$
$n=80$
An 80-gon has an interior angle measure of $175.5^{\circ}$.
6. Is it possible for a regular polygon to have an interior angle measure of $169^{\circ}$ ?

Explain.

$$
\begin{aligned}
& 169^{\circ}=\frac{180^{\circ}(n-2)}{n} \\
& 169^{\circ} n=180^{\circ} n-360^{\circ} \\
& 360^{\circ}=11^{\circ} n \\
& n=32 . \overline{72}
\end{aligned}
$$

Since $n$ is not a natural number, there is no regular polygon with interior angle measure of $169^{\circ}$.
7. As $n$ gets very large, $(n \rightarrow \infty)$, what happens to the measure of an interior angle of a regular polygon? Illustrate this graphically.

$$
\begin{array}{ll}
\begin{array}{l}
\text { Measure of an interior angle of a regular } \\
\text { polygon }
\end{array} & =\frac{180^{\circ}(n-2)}{n} \\
& =\frac{180^{\circ} n}{n}-\frac{360^{\circ}}{n} \\
& =180^{\circ}-\frac{360^{\circ}}{n}
\end{array}
$$

As $n$ gets very large, $(n \rightarrow \infty), \frac{360^{\circ}}{n}$ approaches zero, and the measure of an interior angle approaches $180^{\circ}$.

On the graphing calculator, this appears as asymptotic behavior, with the line $y=$ 180 serving as the horizontal asymptote to the curve $y=\frac{180(x-2)}{x}$.


Participants will now investigate the exterior angles of a polygon. It is often difficult for students to understand intuitively that this sum does not depend on the number of sides of the polygon. Participants may use the polygons that they have already constructed to study exterior angles for the next part of the activity.
8. For the polygons that you created earlier, use a straightedge to extend each side of the polygon as a ray to construct the polygon's exterior angles as follows. Choose a vertex from which to begin, and extend the side to the right, thus making the side into a ray. The angle between the ray just drawn and the consecutive side to the right is $\angle 1$. Continue in a counter-clockwise direction until all exterior angles have been drawn. You will construct as many exterior angles as there are sides of the polygon.

You will now carefully cut and tear the polygon so that each exterior angle keeps its distinct vertex. Cutting from each vertex slightly more than halfway along its corresponding side and then tearing away from the center will accomplish this best. When you have finished, you will have as many pieces as there are exterior angles of the polygon.


Align each exterior angle on the segment below, beginning with the vertex of $\angle 1$ at the arrow and above the line. Using a common vertex (align the vertex of $\angle 2$ with the vertex of $\angle 1$ and the edge of $\angle 2$ to the top edge of $\angle 1$ ), move counterclockwise as shown by the arrow and align $\angle 1$ with $\angle 2$. Continue to align the exterior angles in this manner until all angles have been placed.


If necessary, use the transparencies provided to illustrate this process for participants.
How many complete revolutions do your combined exterior angles make?
The combined exterior angles make one complete revolution.
What is the sum of the measures of the exterior angles of your polygon? Is this true for the polygons of all members of your group?
The sum of the measures of the exterior angles of any polygon is $360^{\circ}$.
9. Complete the chart below, obtaining information from other groups. What conclusion can you draw from the chart?

| Number of sides of <br> the polygon | $\mathbf{4}$ | $\mathbf{5}$ | $\mathbf{6}$ | $\mathbf{7}$ | $\mathbf{8}$ | $\mathbf{9}$ | $\ldots$ | $\boldsymbol{n}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Sum of the measures <br> of the exterior angles <br> (degrees) | 360 | 360 | 360 | 360 | 360 | 360 |  | 360 |

For any polygon, the sum of the measures of the exterior angles is $360^{\circ}$.
10. Complete the following flowchart proof for the sum of the measures of the exterior angles of a triangle.


How may the proof above be extended to prove that the sum of the measures of the exterior angles of an $n$-gon is $360^{\circ}$ ?
Since an interior angle and its related exterior angle form a linear pair at each vertex, there are n linear pairs of angles, totaling 180n degrees. As we discovered previously, the sum of the interior angles of a polygon is 180(n-2). Therefore the exterior angles must have a sum of 180n-[180(n-2)] degrees. Simplifying this expression yields $180 n-(180 n-360)$ degrees or $360^{\circ}$.
11. Determine the measure of an exterior angle of a regular polygon. Complete the table below:

| Number of <br> sides of the <br> regular <br> polygon | $\mathbf{4}$ | $\mathbf{5}$ | $\mathbf{6}$ | $\boldsymbol{7}$ | $\boldsymbol{8}$ | $\boldsymbol{9}$ | $\ldots$ | $\boldsymbol{n}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Sum of the <br> measures of <br> the exterior <br> angles <br> (degrees) | 360 | 360 | 360 | 360 | 360 | 360 | .. | 360 |
| Measure of an <br> exterior angle <br> (degrees) | $\frac{360}{4}=$ | $\frac{360}{5}=$ | $\frac{360}{6}=$ | $\frac{360}{7}=$ | $\frac{360}{8}=$ | $\frac{360}{9}=$ | $\frac{360}{n}$ |  |

12. As the number of sides of the polygon increases, what happens to the measures of each exterior angle of a regular polygon? Explain and illustrate graphically.
As the number of sides of the polygon increases, the measure of each exterior angle approaches zero. This results because the measure of the supplementary interior angle approaches $180^{\circ}$.
On the graphing calculator this appears as asymptotic behavior, with the $x$-axis serving as the horizontal asymptote to the curve $y=\frac{360}{x}$.

|  |  |  |
| :---: | :---: | :---: |

In this activity participants are working at the van Hiele Descriptive Level as they determine properties of polygons, specifically with respect to interior and exterior angles. They approach the Relational Level as they connect with algebra, and the Deductive Level if formal reasoning is used in the flowchart proof.

## Transparency <br> Constructing a Polygon's Exterior Angles



Transparency<br>Sum of the Measures of the Exterior Angles of a Polygon Determining the Sum of a Polygon's Exterior Angles



## Interior and Exterior Angles of a Polygon

1. In your group, draw polygons with 3 to 7 sides. Use a straightedge to draw all the diagonals from one vertex. Share your results to complete the table below.

| Number of <br> sides of the <br> polygon | 3 | 4 | 5 | 6 | 7 | $\ldots$ | $n$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Number of <br> diagonals <br> from one <br> vertex |  |  |  |  |  |  |  |
| Number of <br> triangles <br> created |  |  |  |  |  |  |  |
| Total <br> number of <br> degrees in <br> the |  |  |  |  |  |  |  |
| polygon |  |  |  |  |  |  |  |$\quad$|  |
| :--- | :--- | :--- | :--- | :--- |

2. Using the formula you derived for the sum of the measures of the interior angles of the polygon, complete the table below to determine the measure of an interior angle of each regular polygon.

| Number of <br> sides of the <br> polygon | 3 | 4 | 5 | 6 | 7 | $\ldots$ | $n$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Sum of <br> measures of <br> interior <br> angles <br> (degrees) |  |  |  |  |  |  |  |
| Measure of <br> an interior <br> angle <br> (degrees) |  |  |  |  |  |  |  |

3. If $n=50$, what is the measure of each interior angle of a regular polygon?
4. If $n=100$, what is the measure of each interior angle of a regular polygon?
5. Is it possible for a regular polygon to have an interior angle measure of $175.5^{\circ}$ ? Explain.
6. Is it possible for a regular polygon to have an interior angle measure of $169^{\circ}$ ? Explain.
7. As $n$ gets very large $(n \rightarrow \infty)$ what happens to the measure of an interior angle of a regular polygon? Explain. Illustrate this graphically.
8. For the polygons that you created earlier, use a straightedge to extend each side of the polygon as a ray to construct the polygon's exterior angles as follows. Choose a vertex from which to begin, and extend the side to the right, thus making the side into a ray. The angle between the ray just drawn and the consecutive side to the right is $\angle 1$. Continue in a counter-clockwise direction until all exterior angles have been drawn. You will construct as many exterior angles as there are sides of the polygon.

You will now carefully cut and tear the polygon so that each exterior angle keeps its distinct vertex. Cutting from each vertex slightly more than halfway along its corresponding side and then tearing away from the center will accomplish this best. When you have finished, you will have as many pieces as there are exterior angles of the polygon.


Align each exterior angle on the segment below, beginning with the vertex of $\angle 1$ at the arrow and above the line. Using a common vertex (align the vertex of $\angle 2$ with the vertex of $\angle 1$ and the edge of $\angle 2$ to the top edge of $\angle 1$ ), move counter-clockwise as shown by the arrow and align $\angle 1$ with $\angle 2$. Continue to align the exterior angles in this manner until all angles have been placed.


9. Complete the chart below, obtaining information from other groups. What conclusion can you draw from the chart?

| Number of <br> sides of the <br> polygon | 4 | 5 | 6 | 7 | 8 | 9 | $\cdots$ | $n$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Sum of the <br> measures of <br> the exterior <br> angles <br> (degrees) |  |  |  |  |  |  |  |  |

10. Complete the following flowchart proof for the sum of the measures of the exterior angles of a triangle.


The above is taken from Discovering Geometry: An Inductive Approach: $3{ }^{\text {rd }}$ Edition, ©2003, p. 262, with permission from Key Curriculum Press

How may the proof above be extended to prove that the sum of the measures of the exterior angles of an $n$-gon is $360^{\circ}$ ?
11. Determine the measure of an exterior angle of a regular polygon. Complete the table below:

| Number of <br> sides of the <br> regular polygon | 4 | 5 | 6 | 7 | 8 | 9 | $\ldots$ | $n$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Sum of the <br> measures of the <br> exterior angles <br> (degrees) |  |  |  |  |  |  |  |  |
| Measure of an <br> exterior angle <br> (degrees) |  |  |  |  |  |  |  |  |

12. As the number of sides of the polygon increases, what happens to the measure of each exterior angle of a regular polygon? Explain and illustrate graphically.

## Polygons in Circles

Overview: In this activity, participants determine and apply the area formula for regular polygons and relate properties of regular polygons and their circumscribed circles.
TExES Mathematics Competencies
II.004.A. The beginning teacher Recognizes and extends patterns and
relationships in data presented in tables, sequences, or graphs.
III.011.A. The beginning teacher applies dimensional analysis to
derive units and formulas in a variety of situations (e.g., rates of
change of one variable with respect to another) and to find and
evaluate solutions to problems.
III.013.A. The beginning teacher analyzes the properties of polygons
and their components.
III.013.D. The beginning teacher computes the perimeter, area, and
volume of figures and shapes created by subdividing and combining
other figures and shapes.
V.018.C. The beginning teacher uses inductive reasoning to make
conjectures and uses deductive methods to evaluate the validity of
conjectures.
V.018.D. The beginning teacher uses formal and informal reasoning to
justify mathematical ideas.
V.019.D. The beginning teacher communicates mathematical ideas
using a variety of representations (e.g., numeric, verbal, graphical,
pictorial, symbolic, concrete).

## Geometry TEKS

b.3.D. The student uses inductive reasoning to formulate a conjecture.
b.3.E. The student uses deductive reasoning to prove a statement.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
d.2.C. The student develops and uses formulas including distance and midpoint.
e.1.A. The student finds area of regular polygons and composite figures.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.

Background: Participants need to know the area and perimeter formulas for triangles and circles.

Materials: calculator, centimeter ruler

New Terms: apothem, radius of a polygon

## Procedures:

Participants will derive the formula for the area of a regular polygon in relation to its circumscribed circle. They will also relate properties and parts of inscribed polygons to the corresponding parts and properties of circles.

Each group member should complete the activity for one or more different polygons so that all the polygons are considered. Share your findings with others in your group as you complete the activity.

1. To construct a regular polygon with $n$ sides, connect $n$ equally spaced points on the circumference of a circle. Connect the points on the following congruent circles with a straightedge to complete the regular polygons. Name each polygon.



Heptagon


Octagon


16-gon
2. Draw segments connecting the center of each circle to the vertices of its inscribed polygon. The segments are called the radii of the polygons.

The isosceles triangles formed by the radii and the sides of the polygon can be used to find the area of each polygon. In each polygon draw an altitude to the base of one of the isosceles triangles. This segment is called the apothem of the polygon.

Remind participants to add the new terms apothem and radius of a polygon to their glossaries.

Find the area of a regular polygon in terms of the number of sides, $n$, the length of a side, $s$, and the length of the apothem, $a$.

Example,


Area of polygon $=n($ Area of one triangle $)$
Area of polygon $=n\left(\frac{1}{2} s a\right)$
Area of polygon $=\frac{1}{2} \cdot n s a$
3. Find the area of a regular polygon in terms of its perimeter, $P$, and the length of the apothem, $a$.
$P=n s$
From 2, the area of polygon $=\frac{1}{2} n s a=\frac{1}{2} P a$.
4. Use a ruler to measure, to the nearest millimeter the length of a side and the length of an apothem for each of the regular polygons. Calculate the area of your polygon(s). Complete the following table by sharing your measurements and calculations with members of your group.

| Number of sides <br> $\mathbf{n}$ | Side Length <br> $\mathbf{s}(\mathbf{m m})$ | Apothem <br> $\mathbf{a ( m m})$ | Area <br> $\left(\mathbf{m m}^{\mathbf{2}}\right)$ |
| :---: | :---: | :---: | :---: |
| 3 | 38 | 11 | 627 |
| 4 | 31 | 16 | 992 |
| 5 | 26 | 18 | 1170 |
| 6 | 22 | 19 | 1254 |
| 7 | 19 | 20 | 1330 |
| 8 | 17 | 21 | 1428 |
| 16 | 8 | 22 | 1408 |

The above measurements are for guidance only. Your actual measurements may vary due to variation in the module reproduction processes and measurement error.
5. Based on observation, complete the following statements for polygons and circles with congruent radii.

- As $n$ increases, the length of side $s$ decreases.
- As $n$ increases, the apothem length $a$ increases.
- As $n$ increases, the area of the polygon increases.
- As the chord length $s$ decreases, the distance, $a$, of the chord from the center of the circle increases.
- As the chord length $s$ decreases, the length of its intercepted arc decreases.

As $n$ approaches infinity $(n \rightarrow \infty)$, determine the limits of $P$, the perimeter, and $a$, the apothem. Then determine the limit of the area of the polygon in terms of the limits of $P$ and $a$.

As $n \rightarrow \infty$, the length of the perimeter, $P$, approaches the length of the circumference, $C$, of the circle, and the length of the apothem, $a$, approaches the length of the radius of the circle, $r$.
$P \rightarrow C=2 \pi r$.
Area of polygon $=\frac{1}{2} \mathrm{~Pa}$
Area of polygon $\rightarrow \frac{1}{2} \mathrm{Cr}=\frac{1}{2}(2 \pi r) r=\pi r^{2}$.
The area of a regular polygon approaches the area of its circumscribed circle as the number of sides of the polygon approaches infinity.

Measure the length of the radius in millimeters and determine the area of the circle. The radius measures 22 mm . The area of the circle is approximately $1520 \mathrm{~mm}^{2}$.

Note: For the sample data, the lengths of the apothem for the 16 -gon and for the radius of the circle are 22 mm . Increasing the number of sides of the polygon to numbers greater than sixteen will not change the length of the apothem within measurement tolerance. The factors which affect the area for large values of $n$ are the value of $n$ and the length of $s$. As $n$ increases, $s$ decreases. The area will continue to increase until the limiting value, the area of the circle, is reached.

Participants are performing at the van Hiele Deductive Level in this activity, because properties of polygons and parts of polygons are intrarelated and interrelated with the corresponding properties and parts of circles, and because formulas and conclusions are determined deductively.

## Polygons in Circles

Each group member should complete the activity for one or more different polygons so that all the polygons are considered. Share your findings with others in your group as you complete the activity.

1. To construct a regular polygon with $n$ sides, connect $n$ equally-spaced points on the circumference of a circle. Connect the points on the following congruent circles with a straightedge to complete the regular polygons. Name each polygon.

2. Draw segments connecting the center of each circle to the vertices of its inscribed polygon. The segments are called the radii of the polygons.

The isosceles triangles formed by the radii and the sides of the polygon can be used to find the area of each polygon. In each polygon draw an altitude to the base of one of the isosceles triangles. This segment is called the apothem of the polygon.

Find the area of a regular polygon in terms of the number of sides, $n$, the length of a side, $s$, and the length of the apothem, $a$.
3. Find the area of a regular polygon in terms of its perimeter $P$ and the length of the apothem, $a$.
4. Use a ruler to measure, to the nearest millimeter, the length of a side and the length of an apothem for each of the regular polygons. Calculate the area of your polygon(s). Complete the following table by sharing your measurements and calculations with members of your group.

| Number of sides <br> $\boldsymbol{n}$ | Side Length <br> $\boldsymbol{s}(\mathbf{m m})$ | Apothem Length <br> $\boldsymbol{a}(\mathbf{m m})$ | Polygon Area <br> $\left(\mathbf{m m}^{2}\right)$ |
| :---: | :---: | :---: | :---: |
| 3 |  |  |  |
| 4 |  |  |  |
| 5 |  |  |  |
| 6 |  |  |  |
| 7 |  |  |  |
| 8 |  |  |  |
| 16 |  |  |  |

5. Based on observation, complete the following statements for polygons and circles with congruent radii.

- As $n$ increases, the side length $s$ $\qquad$ .
- As $n$ increases, the apothem length $a$ $\qquad$ .
- As $n$ increases, the area of the polygon $\qquad$ .
- As the chord length $s$ decreases, the distance $a$ of the chord from the center of the circle $\qquad$ .
- As the chord length $s$ decreases, the length of its intercepted arc
$\qquad$ .

As $n$ approaches infinity $(n \rightarrow \infty)$, determine the limits of $P$, the perimeter, and $a$, the apothem. Then determine the limit of the area of the polygon in terms of the limits of $P$ and $a$.

## Angles Associated with a Circle

Overview: This activity illustrates the properties of angles, chords, and tangents of a circle.

Objective: TExES Mathematics Competencies
III.013.B. The beginning teacher analyzes the properties of circles and the lines that intersect them.
V.018.A. The beginning teacher recognizes and uses multiple representations of a mathematical concept (e.g., a point and its coordinates, the area of a circle as a quadratic function of the radius, probability as the ratio of two areas, area of a plane region as a definite integral).
V.018.C. The beginning teacher uses inductive reasoning to make conjectures and uses deductive methods to evaluate the validity of conjectures.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

## Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.D. The student uses inductive reasoning to formulate a conjecture.
b.3.E. The student uses deductive reasoning to prove a statement.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
e.2.C. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of circles and the lines that intersect them.

Background: Participants need to know circle terms such as central and inscribed angle.

Materials: protractor, centimeter ruler
New Terms: cyclic quadrilateral, major arc, minor arc

## Procedures:

Have participants recall the definitions of a central angle and an inscribed angle and that the measure of an inscribed angle is equal to half the measure of the central angle that intercepts the same arc. Also participants should recall what is meant by a secant line and a tangent line to a circle.

We will define two new terms. A minor arc is an arc of a circle that is smaller than a semicircle. A major arc is an arc of a circle that is larger than a semicircle. Remind participants to add these terms to their glossaries.

Participants will use deduction to prove relationships among angles formed by secant and tangent lines to a circle and the associated arcs that are intercepted by these arcs. For that reason many of the solutions are written as proofs.

1. Determine $m F J$ and $m G H$ in circle $A$ below. Explain.

Angle $H$ intercepts $F G$.
Since $m \angle H=40^{\circ}$, then $m F G=80^{\circ}$
Together, FG and GH form a semicircle.
Therefore, $m G H$ is $180^{\circ}-80^{\circ}$, or $100^{\circ}$.
FJ and FH also form a semicircle.
Therefore, $m F J$ is $180^{\circ}-108^{\circ}$, or $72^{\circ}$.

2. In the figure, which is drawn to scale, $\overrightarrow{B A}$ is tangent to circle $C$ at $A ; \overrightarrow{B F}$ is tangent to circle $C$ at $F$. Confirm that $\angle C A B$ and $\angle C F B$ are right angles. Use this fact to prove that $\overline{A B} \cong \overline{F B}$.


Participants may use the corner of a sheet of paper or a protractor to confirm that $\angle C A B$ and $\angle C F B$ are right angles.

| Draw $\overline{B C}$ | (Through any two points a unique <br> line can be drawn.) |
| :--- | :--- |
| $\Delta C F B$ and $\triangle C A B$ are right triangles | (Definition of right triangles) |
| $\overline{A C} \cong \overline{C F}$ | (All radii in a circle are congruent.) |
| $\overline{B C} \cong \overline{B C}$ | (Reflexive Property of Equality) |
| $\Delta C F B \cong \Delta C A B$ | (If the hypotenuse and a leg of one <br> right triangle are congruent to the <br> corresponding hypotenuse and leg <br> of another right triangle, then the <br> triangles are congruent.) |
| $\overline{A B} \cong \overline{F B}$ | (Corresponding sides of congruent <br> triangles are congruent.) |

This problem has two important results that need to be emphasized: 1) the tangent line is perpendicular to the radius at the point of tangency and 2) tangent segments to a circle from a point outside the circle are congruent.
3. $\overrightarrow{G F}$ and $\overrightarrow{G H}$ are tangent to circle $J$. Determine $m \angle F G H$.

Angle GFJ and $\angle G H J$ are right angles. Since the sum of the measures of the interior angles of a quadrilateral is $360^{\circ}, \angle F G H$ and $\angle F J H$ are supplementary.
Therefore, $m \angle F G H=70^{\circ}$.

4. $\overleftrightarrow{A C}$ is tangent to circle $F$ at $A$ and to circle $D$ at $C, \overleftrightarrow{E B}$ is tangent to circles $D$ and $F$ at $E$. Determine $m \angle A B E$ and $m \angle C D E$. Explain.

$m \angle A B E$ and $m \angle C D E$ are equal. $\angle A B E$ and $\angle C D E$ are each angles in two quadrilaterals that have a $78^{\circ}$ angle and two $90^{\circ}$ angles. Since the sum of the measures of the interior angles equals to $360^{\circ}$ in each quadrilateral then,

$$
\begin{array}{ll}
m \angle A B E=360^{\circ}-\left(2 \cdot 90^{\circ}+78^{\circ}\right) \text { and } & m \angle C D E=360^{\circ}-\left(2 \cdot 90^{\circ}+78^{\circ}\right) \\
m \angle A B E=102^{\circ} & m \angle C D E=102^{\circ}
\end{array}
$$

5. Circle $C$ is centered at the origin. $\overrightarrow{A B}$ is tangent to circle $C$ at point $A(-5,12)$. Determine the equation of $\overleftrightarrow{A B}$ in slope-intercept form.


Draw the radius from the origin to the point of tangency, $\bar{A} \overline{C A}$ is perpendicular to $\overleftrightarrow{A B}$. Determine the slope of $\overrightarrow{C A}$.

$$
\begin{aligned}
m & =\frac{12-0}{-5-0} \\
m & =\frac{-12}{5}
\end{aligned}
$$

The slope of tangent $\overleftrightarrow{A B}$ is the negative reciprocal of this slope. Write the equation of $\overleftrightarrow{A B}$ in slope-intercept form. Use point $A(-5,12)$ to determine the y-intercept.

$$
\begin{aligned}
& y=\frac{5}{12} x+b \\
& 12=\frac{5}{12}(-5)+b \\
& 12=\frac{-25}{12}+b \\
& b=\frac{169}{12}
\end{aligned}
$$

The equation of $\overleftrightarrow{A B}$ is $y=\frac{5}{12} x+\frac{169}{12}$.

In 6 and 7 , determine $m \angle A G B$ in terms of $A B$ and $C D$.
6. Hint: Construct $\overline{B D}$. $\angle A D B$ and $\angle C B D$ are inscribed angles.

(The measure of an inscribed angle is half
of the measure of its intercepted arc.)
(The measure of the exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.)
(Substitution)
(Subtraction)
7. Hint: Construct $\overline{B C}$.

$m \angle A C B=\frac{1}{2} \cdot m A B$
$m \angle C B D=\frac{1}{2} \cdot m C D$
$m \angle A G B=m \angle A C B+m \angle D B C$
$m \angle A G B=\frac{1}{2} \cdot m A B+\frac{1}{2} \cdot m C D$
or $\quad m \angle A G B=\frac{1}{2}(m A B+m C D)$
(The measure of an exterior angle of a triangle is equal to the sum of the measures of the remote interior angles.)
(Substitution)
8. $\overline{A G}$ and $\overline{B G}$ are tangent to circle $F$. Determine $m \angle G$ in terms of $A B$ alone and then in terms of $A B$ and $A C B$.

$F B \perp B G ; F A \perp A G$
$\angle A$ and $\angle B$ are right angles
$m \angle A=m \angle B=90^{\circ}$
$m \angle F=m A B$
$m \angle A+m \angle F+m \angle G+m \angle B=360^{\circ}$
$90^{\circ}+m \angle G+90^{\circ}+m A B=360^{\circ}$
$m \angle G=180^{\circ}-m A B$
(A tangent line is perpendicular to the radius at the point of tangency.)
(Definition of perpendicular lines)
(Definition of right angles)
(A central angle has a measure equal to that of its intercepted arc.)
(The sum of the measures of the angles in a quadrilateral is $360^{\circ}$.)
(Substitution Property)
(Subtraction)

This leads to the conclusion that $\angle G$ and $\angle F$ are supplementary.
In addition,

$$
\left.\begin{array}{ll}
m A B=360^{\circ}-m A C B & \text { (The sum of th } \\
\left.360^{\circ} .\right)
\end{array}\right] \begin{array}{ll}
m \angle G=180^{\circ}-\left(360^{\circ}-m A C B\right) & \text { (Substitution) } \\
m \angle G=m A C B-180^{\circ} & \\
m \angle G+m \angle G=m A C B-180^{\circ}+180^{\circ}-m A B & \text { (Addition) } \\
2 \cdot m \angle G=m A C B-m A B & \text { (Division) }
\end{array}
$$

(The sum of the arcs in a circle is
9. $\overleftrightarrow{C D}$ is tangent to circle $F$ at $C$. Determine the relationship between the measures of $\angle B C D$ and $\angle B A C$.


$$
\overline{F C} \perp \overline{C D}
$$

$\angle F C D$ is a right angle
$m \angle F C D=90^{\circ}$
$m \angle F C D=m \angle F C B+m \angle B C D$
$m \angle F C B+\angle B C D=90^{\circ}$
$m \angle F C B=90^{\circ}-m \angle B C D$
$\overline{F C} \cong \overline{F B}$
$\angle F B C \cong \angle F C B$
$m \angle F B C=90^{\circ}-m \angle B C D$
$m \angle F B C+m \angle F C B+m \angle C F B=180^{\circ}$
(A tangent line is perpendicular to the radius at the point of tangency.)
(Definition of perpendicular lines)
(Definition of right angle)
(Angle Addition)
(Substitution)
(Subtraction)
(All radii in a circle are congruent.)
(If two sides of a triangle are congruent, then the angles opposite those sides are congruent.)
(Substitution)
(The sum of the measures of the angles in a triangle is $180^{\circ}$.)
(Substitution)

$$
\begin{aligned}
& m \angle C F B=2 \cdot m \angle B C D \\
& m \angle C F B=m B C \\
& m \angle B A C=\frac{1}{2} \cdot m B C \\
& m B C=2 \cdot m \angle B A C \\
& 2 \cdot m \angle B A C=2 \cdot m \angle B C D \\
& m \angle B A C=m \angle B C D
\end{aligned}
$$

(Subtraction)
(A central angle has a measure equal to that of its intercepted arc.)
(The measure of an inscribed angle is half of the measure of its intercepted arc.)
(Multiplication)
(Substitution)
(Division)

In addition, we can see that $m \angle B C D=\frac{1}{2} \cdot m B C$.
10. $\overline{B G}$ is tangent to circle $D$ at $B$. Determine the measure of $\angle G$ in terms of the intercepted arcs $A B$ and $C B$.


$$
\begin{aligned}
& \angle A B G \cong \angle A C B \\
& m \angle A C B=\frac{1}{2} m A B \text { and } \\
& m \angle B A C=\frac{1}{2} m B C
\end{aligned}
$$

(The angle between a tangent and a chord is congruent to the inscribed angle intercepted by the arc on the same side as the tangent.)
(The measure of an inscribed angle is equal to half of the measure of its intercepted arc.)

$$
\begin{aligned}
& m \angle B A C=m \angle G+m \angle A B G \\
& m \angle B A C=m \angle G+m \angle A C B \\
& \frac{1}{2} m B C=m \angle G+\frac{1}{2} m A B \\
& m \angle G=\frac{1}{2} m B C-\frac{1}{2} m A B= \\
& \frac{1}{2}(m B C-m A B)
\end{aligned}
$$

(The measure of the exterior angle of a triangle is equal to the sum of the measures of the interior remote angles.) (Substitution)
(Substitution)
(Subtraction)
11. A quadrilateral inscribed in a circle is a cyclic quadrilateral. Determine the relationships among the various arcs and inscribed angles.


Draw $\overline{F C}$ and $\overline{F A}$.

$$
\begin{aligned}
& m \angle C D A=\frac{1}{2} m A B C \\
& m \angle A B C=\frac{1}{2} m C D A
\end{aligned}
$$

$$
m A B C+m C D A=360^{\circ}
$$

(The sum of the measures of the arcs in a circle is $360^{\circ}$ )

$$
\begin{array}{ll}
m \angle C D A+m \angle A B C=\frac{1}{2}(m A B C+m C D A) & \text { (Substitution) } \\
m \angle C D A+m \angle A B C=180^{\circ} & \text { (Substitution) } \tag{Substitution}
\end{array}
$$

(The measure of an inscribed angle
and is equal to half of the measure of its intercepted arc.)

Similarly, $m \angle B C D+m \angle B A D=180^{\circ}$

Opposite angles in a cyclic quadrilateral are supplementary.

Determine the relationship between the exterior angle to the diagonally opposite interior angle of a cyclic quadrilateral.


$$
\begin{array}{ll}
m \angle C D A+m \angle A B C=180^{\circ} \text { and } & \text { (Linear pair) } \\
m \angle C D A+m \angle C D K=180^{\circ} & \\
m \angle C D K=m \angle A B C & \text { (Substitution) }
\end{array}
$$

An exterior angle of a cyclic quadrilateral is congruent to the diagonally opposite interior angle.

Remind participants to add the term cyclic quadrilateral to their glossaries.
12. $\overline{A D}$ is a tangent segment to circle $O$ at $A . \overline{A B}$ is a diameter. $\overline{B D}$ intersects the circumference at $F . m A F=80^{\circ} . E$ is a point on $A B$, on the opposite arc to $A F B$, such that $m E B=70^{\circ}$. Find the measures of the marked angles.
$a=\underline{50^{\circ}}$
$b=45^{\circ}$
$c=\underline{50^{\circ}}$
$d=55^{\circ}$
$e=35^{\circ}$
$f=\underline{40^{\circ}}$
$g=55^{\circ}$
$h=50^{\circ}$
$j=\underline{5}^{o}$
$k=\underline{70}^{\circ}$
$l=\underline{75}^{\circ}$
$m=\underline{105^{\circ}}$

One possible starting point is to determine $m A E$ and $m B F$ using diameter $\overline{A B}$. Then, determine the measures of central $\angle B O E$ and inscribed angles $F A B, A F E, B F E$, $A B E, A B F$, and $B E F$. Then, determine $m \angle F A D$. The remaining angles may be determined using the sum of the measures of the interior angles of a triangle as well as the definition of a linear pair.


Participants are performing at the van Hiele Deductive Level in this activity, because they are asked to provide deductive arguments.

## Angles Associated with a Circle

1. Determine $m F J$ and $m G H$ in circle $A$ below. Explain.

2. In the figure, which is drawn to scale $\overrightarrow{B A}$ is tangent to circle $C$ at $A ; \overrightarrow{B F}$ is tangent to circle $C$ at $F$. Confirm that $\angle C A B$ and $\angle C F B$ are right angles. Use this fact to prove that $\overline{A B} \cong \overline{F B}$.

3. $\overrightarrow{G F}$ and $\overrightarrow{G H}$ are tangent to circle $J$. Determine $m \angle F G H$.
4. $\overrightarrow{A C}$ is tangent to circle $F$ at $A$ and to circle $D$ at $C . \overrightarrow{E B}$ is tangent to circles $D$ and $F$ at $E$. Determine $m \angle A B E$ and $m \angle C D E$. Explain.

5. Circle $C$ is centered at the origin. $\overleftrightarrow{A B}$ is tangent to circle $C$ at point $A(-5$, 12). Determine the equation of $\overrightarrow{A B}$ in slope-intercept form.

In 6 and 7, determine $m \angle A G B$ in terms of $A B$ and $C D$.
6. Hint: Construct $\overline{B D}$.

7. Hint: Construct $\overline{B C}$.

8. $\overline{A G}$ and $\overline{B G}$ are tangent to circle $F$. Determine $m \angle G$ in terms of $A B$ alone and then in terms of $A B$ and $A C B$.

9. $\overrightarrow{C D}$ is tangent to circle $F$ at $C$. Determine the relationship between the measures of $\angle B C D$ and $\angle B A C$.

10. $\overline{B G}$ is tangent to circle $D$ at $B$. Determine the measure of $\angle G$ in terms of the intercepted arcs $A B$ and $C B$.

11. A quadrilateral inscribed in a circle is a cyclic quadrilateral. Determine the relationships among the various arcs and inscribed angles.

12. $\overline{A D}$ is a tangent segment to circle $O$ at $A . \overline{A B}$ is a diameter. $\overline{B D}$ intersects the circumference at $F . m A F=80^{\circ} . E$ is a point on $A B$, on the opposite arc to $A F B$, such that $m E B=70^{\circ}$. Find the measures of the marked angles.

$$
\begin{array}{lllll}
a=\ldots & b=\ldots & c=\ldots & d=\ldots & e=\ldots \\
g=\ldots & h=\ldots & j=\ldots & k=\ldots & l=\ldots
\end{array} \quad m=\ldots
$$



## Parts of a Circle

Overview: In this activity, participants determine the areas of sectors as proportional parts of the area of the whole circle, areas of segments as parts of the areas of sectors, and the areas of annuli as the difference between concentric circles.

Objective: TExES Mathematics Competencies
III.011.A. The beginning teacher applies dimensional analysis to derive units and formulas in a variety of situations (e.g., rates of change of one variable with respect to another) and to find and evaluate solutions to problems.
III.011.D. The beginning teacher applies the Pythagorean theorem, proportional reasoning, and right triangle trigonometry to solve measurement problems.
III.012.B. The beginning teacher uses properties of points, lines, planes, angles, lengths, and distances to solve problems.
III.013.B. The beginning teacher analyzes the properties of circles and the lines that intersect them.
III.013.D. The beginning teacher computes the perimeter, area, and volume of figures and shapes created by subdividing and combining other figures and shapes.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

## Geometry TEKS

b.3.B. The student constructs and justifies statements about geometric figures and their properties.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
d.2.C. The student develops and uses formulas including distance and midpoint.
e.1.A. The student finds area of regular polygons and composite figures.
e.1.B. The student finds area of sectors and arc lengths of circles using proportional reasoning.
e.2.C. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of circles and the lines that intersect them.
f.2. The student uses ratios to solve problems involving similar figures.

Background: Participants need to know circle terms, such as the various angles of a circle, properties of chords, and tangent properties, and the formula for the area of a circle.

Materials: compass, centimeter ruler, calculator, easel paper, colored markers
New Terms: annulus (plural is annuli), sector of a circle, segment of a circle

## Procedures:

Participants will define and determine methods for finding areas of sectors, segments of circles, and annuli. Participants complete $1-5$ in their groups (about ten minutes). During whole-group discussion, ask participants to share their strategies for completing $1-5$. The remaining items may be completed in groups and presented on easel paper if desired. These problems are also suitable for a homework assignment.

1. The shaded region in the circle is called a sector. Define sector of a circle.


A sector is the region bounded by an arc and the two radii to the endpoints of the arc.
2. The length of the arc and the area of the sector may be related to the length of the circumference and the area of the circle, respectively. Find the length of the arc and the area of the sector in terms of the radius of the circle, $r$, and the central angle, $\theta$, measured in degrees, which determines the sector.
The ratio of the measure of the central angle to $360^{\circ}$ is equal to the ratio of the arc length to the length of the circumference, and also to the ratio of the area of the sector to the area of the circle.

$$
\frac{\theta}{360^{\circ}}=\frac{\text { arc length }}{2 \pi r}=\frac{\text { area of sector }}{\pi r^{2}}
$$

Arc length $\quad=\quad \frac{\theta}{360^{\circ}} 2 \pi r$.
Area of sector $=\frac{\theta}{360^{\circ}} \pi r^{2}$.
3. The shaded region is a segment of a circle. Define segment of a circle.


A segment of a circle is the region between a chord of a circle and the included arc.

Explain how to find the area of the segment of a circle.
The area of a segment can be determined by subtracting the area of the triangle bounded by the two radii and the chord from the area of the sector.
4. Determine the areas of the shaded segment and the sector for each circle.


Circle A:
Area of sector $=\frac{90^{\circ}}{360^{\circ}} \pi \cdot 5^{2}=\frac{25 \pi}{4} \mathrm{~cm}^{2}$.
Area of segment $=$ Area of sector - Area of triangle.
Area of segment $=\frac{25 \pi}{4}-\frac{25}{2} \mathrm{~cm}^{2}$.


## Circle B:

Area of sector $=\frac{120^{\circ}}{360^{\circ}} \pi \cdot 7^{2}=\frac{49 \pi}{3} \mathrm{~cm}^{2}$.
Area of segment $=$ Area of sector - Area of triangle.
Area of triangle $=\frac{7}{2}\left(\frac{7}{2} \sqrt{3}\right)=\frac{49}{4} \sqrt{3} \mathrm{~cm}^{2}$.
Area of segment $=\frac{49 \pi}{3}-\frac{7}{2}\left(\frac{7}{2} \sqrt{3}\right)$
$=\frac{49 \pi}{3}-\frac{49 \sqrt{3}}{4} \mathrm{~cm}^{2}$.
5. The cross section of a doughnut or bagel is a real-world example of an annulus (plural, annuli). Using your compass, draw an annulus. Label the radius of the smaller (inner) circle $r$ and the radius of the larger (outer) circle R. Define annulus. Explain how to find the area of an annulus.


## An annulus is the region bounded by two concentric circles.

Area of annulus $=\pi R^{2}-\pi r^{2}$
Area of annulus $=\pi\left(R^{2}-r^{2}\right)$.

Remind participants to add the new terms sector of a circle, segment of a circle, and annulus to their glossaries.
6. Determine the area of an annulus with the radius, $r$, of the smaller circle 6 cm and the radius, $R$, of the larger circle 10 cm .

$$
\begin{aligned}
\text { Area of annulus } & =\pi\left(R^{2}-r^{2}\right) \\
& =\pi\left(10^{2}-6^{2}\right) \\
& =64 \pi \mathrm{~cm}^{2} .
\end{aligned}
$$

7. Determine the measure of the central angle in the shaded region given that the area of the shaded sector is $15 \pi \mathrm{~cm}^{2}$ and the radius is 6 cm .

$$
\begin{aligned}
\text { Area of sector } & =\frac{\theta}{360} \cdot \pi r^{2} \\
15 \pi & =\frac{\theta}{360} \cdot \pi 6^{2} \\
15 \pi & =\frac{\pi \theta}{10} \\
\theta & =150^{\circ}
\end{aligned}
$$


8. Determine $r$, the radius of the smaller circle, given the shaded area is $30 \pi \mathrm{~cm}^{2}$ and the radius of the larger circle is 18 cm .
Area of annulus $=\pi R^{2}-\pi r^{2}$

$$
\begin{aligned}
30 \pi & =\pi 18^{2}-\pi r^{2} \\
30 \pi & =324 \pi-\pi r^{2} \\
\pi r^{2} & =294 \pi \\
r & \approx 17.15 \mathrm{~cm}
\end{aligned}
$$


9. Determine the radius of each circle given that the area of the shaded region is $15 \pi$

$$
\begin{aligned}
& \mathrm{cm}^{2} \text { and } r=\frac{2}{3} R . \\
& \text { Area of annulus }=\frac{5}{6}\left(\pi R^{2}-\pi r^{2}\right) \\
& \begin{aligned}
150 \pi \quad & =\frac{5 \pi}{6}\left[R^{2}-\left(\frac{2}{3} R\right)^{2}\right] \\
180 & =R^{2}-\frac{4 R^{2}}{9} \\
180 & =\frac{5 R^{2}}{9} \\
324 & =R^{2} \\
R \quad & =18 \mathrm{~cm} \\
r \quad & =\frac{2}{3} \cdot 18 \mathrm{~cm} \\
r \quad & =12 \mathrm{~cm}
\end{aligned}
\end{aligned}
$$

10. In the figure, $\overline{U S}$ is a chord of the larger concentric circle and tangent to the smaller concentric circle. $U S=20 \mathrm{~cm}$. Find the area of the annulus.


Chord $\overline{U S}$ is bisected at the point of tangency, since $\Delta S T U$ is an isosceles triangle.
Area of annulus $=\pi\left(R^{2}-r^{2}\right)$.
Using the Pythagorean Theorem,
$R^{2}-r^{2}=10^{2}$ or $R^{2}-r^{2}=100$.
Area of annulus $=100 \pi \mathrm{~cm}^{2}$.
11. A circle circumscribes a hexagon with side length of 8 meters. Find the area of the region between the hexagon and the circle.


Area of shaded region $=$ Area of circle - Area of hexagon.

$$
\begin{aligned}
\text { Area of shaded region } & =\pi r^{2}-\frac{1}{2} P a \\
& =64 \pi-\frac{1}{2} 48(8 \sqrt{3}) \\
& =64 \pi-192 \sqrt{3} \mathrm{~m}^{2} .
\end{aligned}
$$

12. Circle $O$ is inscribed in the square $A B C D$ with sides of length 12 inches. The diameters shown are perpendicular bisectors of $\overline{A B}$ and $\overline{A D}$. Find the area of the unshaded region inside the square.


The length of the radius of $O$ is 6 in .

$$
\text { Shaded area }=\text { Area of semi-circle }- \text { Area of } \triangle P Q R
$$

$$
\begin{aligned}
\text { Shaded area } & =\frac{\pi \cdot 6^{2}}{2}-\frac{1}{2}(12)(3) \\
& =18 \pi-18 \mathrm{in}^{2}
\end{aligned}
$$

$$
\begin{aligned}
\text { Area of square } A B C D & =(12)^{2}=144 \mathrm{in}^{2} \\
\text { Area of unshaded region } & =144-(18 \pi-18) \\
& =162-18 \pi \mathrm{in}^{2} .
\end{aligned}
$$

Participants are performing at the van Hiele Relational Level in this activity since they are required to select from the various properties and formulas among different figures in order to solve the problems.

## Parts of a Circle

1. The shaded region in the circle is called a sector. Define sector of a

2. The length of the arc and the area of the sector may be related to the length of the circumference and the area of the circle respectively. Find the length of the arc and the area of the sector in terms of the radius of the circle, $r$, and the central angle, $\theta$, measured in degrees, which determines the sector.
3. The shaded region is a segment of a circle. Define segment of a circle.


Explain how to find the area of the segment of a circle.
4. Determine the areas of the shaded segment and the sector for each circle.

5. The cross section of a doughnut or bagel is a real-world example of an annulus (plural, annuli). Using your compass, draw an annulus. Label the radius of the smaller (inner) circle $r$ and the radius of the larger (outer) circle $R$. Define annulus. Explain how to find the area of an annulus.
6. Determine the area of an annulus with the radius, $r$, of the smaller circle 6 cm and the radius, $R$, of the larger circle 10 cm .
7. Determine the measure of the central angle in the shaded region given that the area of the shaded sector is $15 \pi \mathrm{~cm}^{2}$ and the radius is 6 cm .

8. Determine $r$, the radius of the smaller circle, given the shaded area is 30 $\pi \mathrm{cm}^{2}$ and the radius of the larger circle is 18 cm .

9. Determine the radius of each circle given that the area of the shaded region is $15 \pi \mathrm{~cm}^{2}$ and $r=\frac{2}{3} R$.

10. In the figure, $\overline{U S}$ is a chord of the larger concentric circle and tangent to the smaller concentric circle. $U S=20 \mathrm{~cm}$. Find the area of the annulus.

11. A circle circumscribes a hexagon with side length of 8 meters. Find the area of the region between the hexagon and the circle.

12. Circle $O$ is inscribed in the square $A B C D$ with sides of length 12 inches. The diameters shown are perpendicular bisectors of $\overline{A B}$ and $\overline{A D}$. Find the area of the unshaded region inside the square.


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## Unit 8 - Similarity and Trigonometry Magnification Ratio

Overview: In this activity participants use coordinate geometry, distance, angle measure, and inductive reasoning to investigate the attributes of similar figures to formalize the definition of similar figures.

Objective: TExES Mathematics Competencies
III.011.B. The beginning teacher applies formulas for perimeter, area, surface area, and volume of geometric figures and shapes (e.g., polygons, pyramids, prisms, cylinders, cones, spheres) to solve problems.
III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.014.E. The beginning teacher applies concepts and properties of slope, midpoint, parallelism, perpendicularity, and distance to explore properties of geometric figures and solve problems in the coordinate plane.
V.018.B. The beginning teacher applies correct mathematical reasoning to derive valid conclusions from a set of premises. V.018.D. The beginning teacher uses inductive reasoning to make conjectures and uses deductive methods to evaluate the validity of conjectures.

## Geometry TEKS

b.1.A. The student develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems.
b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.D. The student uses inductive reasoning to formulate a conjecture.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
d.2.A. The student uses one- and two- dimensional coordinate systems to represent points, lines, line segments, and figures.
d.2.B. The student uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons.
d.2.C. The student develops and uses formulas including distance and midpoint.
e.1.A. The student finds areas of regular polygons and composite figures.
f.1. In a variety of ways, the student develops, applies, and justifies triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples.

Background: Participants should have some experience with translations on the coordinate plane prior to this activity.

Materials: graphing calculator, compass, centimeter grid paper, protractor, patty paper, straightedge

## New Terms:

## Procedures:

Distribute the activity page to each participant. Engage participants in a brief discussion about congruent figures using the following questions.

What are congruent figures?
Responses should include figures that have the same shape and size.
Are the two figures in 1 congruent? Explain.
No. The two figures are the same shape with congruent corresponding angles; however, corresponding sides are not the same length.

Participants will develop the formal definition of similar polygons. Two polygons are similar if and only if the corresponding angles are congruent and corresponding sides are proportional.

Use the activity to explore attributes of geometric figures to create a definition for similar polygons.

Ask participants to list the geometric features of the two figures drawn in 1.

1. Trace $\triangle A B C$ onto a piece of patty paper. Place the tracing of $\triangle A B C$ over $\triangle D E F$ so that $\angle A$ corresponds to $\angle D$. Describe what you observe about the two triangles. Repeat this process for the remaining angles.
The corresponding angles are congruent, and the side lengths of $\triangle D E F$ appear to be twice as long as the side lengths of $\triangle A B C$.

2. Use coordinate geometry to determine and describe the relationships between the two triangles above.

Location of triangle vertices:
$A(1,1) ; B(2,1) ; C(2,3) ; D(3,2) ; E(5,2) ; F(5,6)$

- The $x$-coordinate of each vertex in $\triangle D E F$ is 2 times the corresponding $x$ coordinate in $\triangle A B C$ and translated one unit to the right.
- The y-coordinate of each vertex in $\triangle D E F$ is 2 times the corresponding $y$ coordinate in $\triangle A B C$.

Segment lengths:
$A B=1 ; B C=2 ; A C=\sqrt{5} ; D E=2 ; E F=4$;
$D F=\sqrt{(5-3)^{2}+(6-2)^{2}}=\sqrt{20}=2 \sqrt{5}$
$\overline{D E}$ is 2 times longer than $\overline{A B}$.

- $\overline{E F}$ is 2 times longer than $\overline{B C}$.
- $\overline{D F}$ is 2 times longer than $\overline{D E}$.
- The lengths of the sides of $\triangle D E F$ are 2 times longer than the lengths of the sides of $\triangle A B C$.
- $\triangle D E F$ is an enlargement of $\triangle A B C$ by a factor of 2. $\triangle A B C$ has been magnified 2 times to produce $\triangle D E F$.

Slopes of corresponding sides:
Slope of $\overline{A C}=\frac{3-1}{2-1}=2$
Slope of $\overline{C B}=$ undefined
Slope of $\overline{A B}=0$
The slopes of corresponding sides of $\triangle A B C$ and $\triangle D E F$ are equal.

Coordinate rule or mapping of corresponding vertices:

$$
\begin{aligned}
& A(x, y) \rightarrow D(2 x+1,2 y) \\
& B(x, y) \rightarrow E(2 x+1,2 y) \\
& C(x, y) \rightarrow F(2 x+1,2 y)
\end{aligned}
$$

Angle Measures:

$$
\begin{aligned}
& m \angle C A B \approx 63^{\circ} \mathrm{m} \angle F D E \approx 63^{\circ} \\
& m \angle A B C \approx 90^{\circ} \mathrm{m} \angle D E F \approx 90^{\circ} \\
& m \angle B C A \approx 27^{\circ} \mathrm{m} \angle E F D \approx 27^{\circ}
\end{aligned}
$$

Corresponding angles of $\triangle A B C$ and $\triangle D E F$ are congruent.

Perimeter:
Perimeter of $\triangle A B C$ is equal to $3+\sqrt{5}$
Perimeter of $\triangle D E F$ is equal to $6+2 \sqrt{5}=2(3+\sqrt{5})$
The perimeter of $\triangle D E F$ is 2 times as long as the perimeter of $\triangle A B C$. The perimeter of $\triangle A B C$ has been magnified 2 times.

Area:
Area of $\triangle A B C$ is equal to 1 square unit
Area of $\triangle D E F$ is equal to 4 square units
The area of $\triangle D E F$ is the square of the magnification ratio, 4 times larger than the area of $\triangle A B C$.
3. Draw any polygon on the coordinate plane below. Select a magnification ratio. Draw a second polygon that preserves the shape of the original polygon but whose size is determined by the magnification ratio you selected.
Ask participants to draw different polygons.
4. Describe the relationships between the pairs of corresponding sides, perimeters of the polygons, coordinate points of corresponding vertices, measures of corresponding angles, and areas of the two polygons you drew in 3.
Responses should reflect the discussion similar to 2 above.
5. Compare results within your group and make a conjecture about similarity for geometric shapes.
Responses will vary and may include:

- Similar polygons are polygons whose corresponding sides are proportional and whose corresponding angles are congruent.
- Similarity is a transformation that preserves angles and changes all distances in the same ratio. This ratio is referred to as a ratio of magnification or a scale factor.
- Congruency is a special case of similarity when the scale factor is one.

6. Describe $\Delta G H J$ if $\triangle A B C$ is mapped to $\triangle G H J$ according to the coordinate mapping of corresponding vertices:
$A(x, y) \rightarrow G(1.5 x+2,1.5 y)$
$B(x, y) \rightarrow H(1.5 x+2,1.5 y)$
$C(x, y) \rightarrow J(1.5 x+2,1.5 y)$

- $\triangle G H J$ is an enlargement of $\triangle A B C$ by a scale factor of 1.5. $\triangle A B C$ has been magnified 1.5 times to produce $\Delta G H J$.
- This transformation is a result of increasing the side lengths of $\triangle A B C$ by a scale factor of 1.5 and adding two units to the $x$-coordinates.


7. What information is required in order to prove that two triangles are similar? In order to prove that two triangles are similar one must prove that the corresponding angles of the triangles are congruent and that the corresponding sides of the triangles are proportional. There are some shortcuts that can be used.

Have participants complete the three investigations for shortcuts to determine when two triangles are similar. Participants should work in groups of four or five. Each group should complete the steps in the investigation with a different triangle. Group members should share their results to arrive at three results.

The following is taken from Discovering Geometry: An Investigative Approach, $3^{\text {rd }}$ Edition, © 2003, p. 572-574 with permission from Key Curriculum Press.
8. Investigation 1

Is AA a Similarity Shortcut?
Step 1 Draw any triangle $A B C$.
Step 2 Construct a second triangle, $D E F$ with $\angle D \cong \angle A$ and $\angle E \cong \angle B$. What will be true about $\angle C$ and $\angle F$ ? Why?
This step reviews the Triangle Sum Theorem. It is why AAA can be reduced to AA. $\angle F$ must be congruent to $\angle C$ because three angles of each triangle must add up to $180^{\circ}$.

Step 3 Carefully measure the lengths of the sides of both triangles. Compare the lengths of the corresponding sides. Is $\frac{A B}{D E} \approx \frac{A C}{D F} \approx \frac{B C}{E F}$ ?
The approximation symbols are important. Measurement error, and any conversion to decimals, might yield slightly unequal ratios. The ratios should be equal.

Step 4 Compare the results of others near you. You should be ready to state a theorem.

## AA Similarity Theorem

If $\underline{2}$ angles of one triangle are congruent to $\underline{2}$ angles of another triangle then the triangles are similar.

Note: There is no need to investigate the AAA Similarity Theorem, since it is equivalent to the AA Similarity Theorem.
9. Investigation 2

Is SSS a Similarity Shortcut?
If three sides of one triangle are proportional to three sides of another triangle, must the two triangles be similar?

Draw any triangle $A B C$. Then construct a second triangle $D E F$, whose side lengths are a multiple of the original triangle. (Your second triangle may be larger or smaller than $\triangle A B C$.)

Compare the corresponding angles of the two triangles. Compare your results with the results of others near you and state a theorem.
Corresponding angles should be congruent.

## SSS Similarity Theorem

If the three sides of one triangle are proportional to the three sides of another triangle, then the triangles are similar.
10. Investigation 3

Is SAS a Similarity Shortcut?
Is SAS a shortcut for similarity? Try to construct two different triangles that are not similar but have two pairs of sides proportional and the pair of included angles equal in measure.
Corresponding sides should be proportional, and corresponding angles congruent.
Compare the measures of the corresponding sides and the measures of the corresponding angles. Share your results with others near you and state a theorem.

## SAS Similarity Theorem

If two sides of one triangle are proportional to two corresponding sides of another triangle and the included angles are congruent then the triangles are similar.

To summarize:

- Angle - Angle Similarity Theorem or AA Similarity Theorem: If two angles of one triangle are congruent to two angles of another triangle then the triangles are similar.
If two angles of a triangle are congruent to two angles of another triangle then the third angles of the triangles are congruent. Since we know that two triangles are similar by the AAA Similarity Theorem we can now say that two triangles are similar when we know that they have two angles congruent.
- Side - Side - Side Similarity Theorem or SSS Similarity Theorem: If three sides of one triangle are proportional to the three sides of another triangle, then the triangles are similar.
- Side - Angle - Side Similarity Theorem or SAS Similarity Theorem: If two sides of one triangle are proportional to two sides of another triangle and the included angles are congruent, then the triangles are similar.

11. The following is taken from Discovering Geometry: An Investigative Approach, $3^{\text {rd }}$ Edition, © 2003, p. 583 with permission from Key Curriculum Press.
While vacationing in Egypt, the Greek Mathematician Thales calculated the height of the Great Pyramid. According to legend, Thales placed a pole at the tip of the pyramid's shadow and used similar triangles to calculate its height. This involved some estimating since he was unable to measure the distance from directly beneath the height of the pyramid to the tip of the shadow. From the diagram, explain his method. Calculate the height of the pyramid from the information given in the diagram.


Thales used similar right triangles to calculate the height of the pyramid. The height of the pyramid and 240 m are the lengths of the legs of one triangle: 6.2 m and 10 m are the lengths of the corresponding legs of the other triangle.
Solving the proportion $\frac{6.2 \mathrm{~m}}{10 \mathrm{~m}}=\frac{H}{240 \mathrm{~m}}$
results in $\quad 10 \mathrm{~m} \cdot \mathrm{H}=6.2 \mathrm{~m} \cdot 240 \mathrm{~m}$
or

$$
H=\frac{1488 m^{2}}{10 m}
$$

$$
H=148.8 m
$$

In this activity participants are working at the Descriptive Level in the first part of the activity since measurement and inductive comparison methods are used. Participants approach the Relational Level in identifying and applying the properties of similar figures in an unusual problem.

## Magnification Ratio

Explore the attributes of geometric figures to create a definition for similar polygons.

1. Trace $\triangle A B C$ onto a piece of patty paper. Place the tracing of $\triangle A B C$ over $\triangle D E F$ so that $\angle A$ corresponds to $\angle D$. Describe what you observe about the two triangles. Repeat this process for the remaining angles.

2. Use coordinate geometry to determine and describe the relationships between the two triangles above.
3. Draw any polygon on the coordinate plane below. Select a magnification ratio. Draw a second polygon that preserves the shape of the original polygon but whose size is determined by the magnification ratio you selected.

4. Describe the relationships between the pairs of corresponding sides, perimeters of the polygons, coordinate points of corresponding vertices, measures of corresponding angles, and areas of the two polygons you drew in 3.
5. Compare results with your group and make a conjecture about similarity for geometric shapes.
6. Describe $\triangle G H J$ if $\triangle A B C$ is mapped to $\triangle G H J$ according to the coordinate mapping of corresponding vertices:
$A(x, y) \rightarrow G(1.5 x+2,1.5 y)$
$B(x, y) \rightarrow H(1.5 x+2,1.5 y)$
$C(x, y) \rightarrow J(1.5 x+2,1.5 y)$

7. What information is required in order to prove that two triangles are similar?
8. Investigation 1

Is AA a Similarity Shortcut?
Step 1 Draw any triangle ABC.
Step 2 Construct a second triangle, DEF with $\angle D \cong \angle A$ and $\angle E \cong \angle B$ . What will be true about $\angle C$ and $\angle F$ ? Why?

Step 3 Carefully measure the lengths of the sides of both triangles. Compare the lengths of the corresponding sides. Is $\frac{A B}{D E} \approx \frac{A C}{D F} \approx \frac{B C}{E F}$ ?

Step 4 Compare the results of others near you. You should be ready to state a theorem.

## AA Similarity Theorem

If $\qquad$ angles of one triangle are congruent to $\qquad$ angles of another triangle then $\qquad$ .

There is no need to investigate the AAA Similarity Theorem, since it is equivalent to the AA Similarity Theorem.
9. Investigation 2

Is SSS a Similarity Shortcut?
If three sides of one triangle are proportional to three sides of another triangle, must the two triangles be similar?

Draw any triangle $A B C$. Then construct a second triangle $D E F$, whose side lengths are a multiple of the original triangle. (Your second triangle may be larger or smaller than $\triangle A B C$.)

Compare the corresponding angles of the two triangles. Compare your results with the results of others near you and state a theorem.

## SSS Similarity Theorem

If the three sides of one triangle are proportional to the three sides of another triangle, then the triangles are $\qquad$ .
10. Investigation 3

Is SAS a Similarity Shortcut?
Is SAS a shortcut for similarity? Try to construct two different triangles that are not similar but have two pairs of sides proportional and the pair of included angles equal in measure.

Compare the measures of the corresponding sides and the measures of the corresponding angles. Share your results with others near you and state a theorem.

## SAS Similarity Theorem

If two sides of one triangle are proportional to two corresponding sides of another triangle and $\qquad$ then the $\qquad$ .
11. While vacationing in Egypt, the Greek Mathematician Thales calculated the height of the Great Pyramid. According to legend, Thales placed a pole at the tip of the pyramid's shadow and used similar triangles to calculate its height. This involved some estimating since he was unable to measure the distance from directly beneath the height of the pyramid to the tip of the shadow. From the diagram, explain his method. Calculate the height of the pyramid from the information given in the diagram.


## What Do You Mean?

Overview: In this activity, participants use their knowledge of similarity to formulate conjectures about the proportional relationships in the right triangles formed when the geometric mean is constructed.

## Objective: TExES Mathematics Competencies

III.012.E. The beginning teacher describes and justifies geometric constructions made using compass and straightedge, reflection devices, and other appropriate technologies.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles). III.014.C. The beginning teacher uses transformations to explore and describe reflectional, rotational, and translational symmetry.
V.018.B. The beginning teacher applies correct mathematical reasoning to derive valid conclusions from a set of premises.
V.018.C. The beginning teacher uses inductive reasoning to make conjectures and uses deductive methods to evaluate the validity of conjectures.

## Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships. b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.D. The student uses inductive reasoning to formulate a conjecture.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
d.2.A. The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.
d.2.B. The student uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons.
d.2.C. The student develops and uses formulas including distance and midpoint.
f.3. In a variety of ways, the student develops, applies, and justifies triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples.

Background: Participants should have reviewed proportional relationships in similar figures.

Materials: compass, centimeter grid paper, patty paper, centimeter ruler
New Terms: geometric mean

## Procedures:

1. Guide participants through the construction of a geometric mean.

Step 1 Construct a circle with center $(0,0)$ on a coordinate grid.


Step 3 Construct a perpendicular through $D$ on the diameter. $D$ should have an integer value $x$-coordinate that is not the center of the circle. Label the intersection of the perpendicular and the circle with a positive $y$-coordinate, $B$.

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| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $B$ |  |  | - |  |  |  |  |
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|  |  |  |  | ${ }^{2}$ |  |  |  |  |
|  |  |  |  | - |  |  |  |  |
| A | $D$ |  |  |  |  |  |  | $C$ |
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## What kind of triangle is $\triangle A B C$ ?

The triangle is a right triangle.

Step 2 Construct a diameter, $\overline{A C}$, of the circle along the $x$-axis.


Step 4 Construct segments from the endpoints of the diameter to one of the intersections, $B$, of the perpendicular and the circle.


Step 5 Determine $A B, B C, A D, D C$, and $A C$. Use the radius to determine the exact coordinates of $B$.


In this example,
Coordinates of $B$ are $(-2,2 \sqrt{3})$
$B D=\sqrt{C B^{2}-C D^{2}}=\sqrt{4^{2}-2^{2}}=\sqrt{12}=2 \sqrt{3}$
$A D=2, D C=6, A C=8, A B=\sqrt{(-4-(-2))^{2}+(0-2 \sqrt{3})^{2}}=4$,
$B C=\sqrt{(4-(-2))^{2}+(0-2 \sqrt{3})^{2}}=4 \sqrt{3}$
Note that $\overline{B D}$ is an altitude of $\triangle A B C$.
Step 6 - Trace the triangles onto another sheet of paper and cut apart $\triangle A B D$ and $\triangle D B C$. Cut out the original $\triangle A B C$ and determine whether the three triangles are similar. Justify your answer.

$\triangle A B D$ and $\triangle B C D$ can be rotated and translated to show $\angle A B D \cong \angle B C D, \angle D A B \cong \angle D B C$, and $\angle B D A \cong \angle C D B$. Using the lengths that were calculated earlier, $\triangle A B D$ and $\triangle B C D$ are similar with a scale factor of $\sqrt{3} . \triangle B C D$ and $\triangle A C B$ can be rotated and translated to show $\angle C B D \cong \angle C A B, \angle B D C \cong \angle A B C$, and $\angle D C B \cong \angle B C A$. Using the lengths that were calculated earlier, $\triangle B C D$ and $\triangle A C B$ are similar with a scale factor of $\frac{2 \sqrt{3}}{3}$.
$\triangle A B D$ and $\triangle A C B$ can be rotated and translated to show $\angle D A B \cong \angle B A C, \angle A D B \cong \angle A B C$, and $\angle A B D \cong \angle A C B$.

Using the lengths that were calculated earlier, $\triangle A B D$ and $\triangle A C B$ are similar with a scale factor of 2 .
$\therefore \triangle A B D \sqcup \triangle B C D \sqcup \triangle A C B$.
2. Describe the proportional relationship that includes $\overline{B D}$ in $\triangle A B D$ and $\triangle B C D$.

$\overline{A D}$ in $\triangle A B D$ corresponds to $\overline{B D}$ in $\triangle B C D . \overline{D B}$ in $\triangle A B D$ corresponds to $\overline{D C}$ in $\triangle B C D$.
From the proportion $\frac{A D}{B D}=\frac{B D}{D C}, B D=\sqrt{A D \cdot D C}$

Explain to participants that $B D$ is the geometric mean of $A D$ and $D C$. The definition of the geometric mean is as follows: the number $b$ is the geometric mean between the numbers $a$ and $c \Leftrightarrow a, b, c$ are positive and $\frac{a}{b}=\frac{b}{c}$ or $b=\sqrt{a c}$.
Remind participants to add the new term geometric mean to their glossaries.

## How is the arithmetic mean determined?

Find the sum of the terms and divide by the number of terms.
3. How can the construction in 1 be altered so that $\overline{B D}$ represents the arithmetic mean? Construct $\overline{B D}$ so that it lies on the perpendicular bisector of $\overline{A C}$.
This relationship can be shown algebraically as follows:

$$
\begin{aligned}
& \text { If } B D=\sqrt{A D \cdot D C} \text { and } B D=\frac{A D+D C}{2} \\
& \text { then } \sqrt{A D \cdot D C}=\frac{A D+D C}{2} \\
& A D \cdot D C=\left(\frac{A D+D C}{2}\right)^{2} \\
& A D \cdot D C=\frac{A D^{2}+2 A D \cdot D C+D C^{2}}{4} \\
& 4 A D \cdot D C=A D^{2}+2 A D \cdot D C+D C^{2} \\
& 0=A D^{2}-2 A D \cdot D C+D C^{2} \\
& 0=(A D-D C)^{2} \\
& 0=A D-D C \\
& A D=D C
\end{aligned}
$$

$\therefore D$ is the midpoint of $\overline{A C}$.
4. What type of triangle is formed when the arithmetic mean is constructed?

An isosceles right triangle is formed by the construction.
5. Describe the point on the hypotenuse where the arithmetic mean is equal to the geometric mean of the lengths of the segments formed.
The midpoint of the hypotenuse satisfies this condition.
Participants successful with this activity are working at the van Hiele Relational Level because they interrelate properties of right triangles, including the Pythagorean Theorem, with properties of similar figures. In 3 participants justify a relationship using deductive algebraic methods, which requires Deductive Level ability.

## What Do You Mean?

1. Construct a geometric mean.

Step 1 Construct a circle with center $(0,0)$ on a coordinate grid.
Step 2 Construct a diameter, $\overline{A C}$, of the circle along the $x$-axis.
Step 3 Construct a perpendicular through $D$ on the diameter. $D$ should have an integer value $x$-coordinate that is not the center of the circle. Label the intersection of the perpendicular and the circle with a positive $y$-coordinate, $B$.
Step 4 Construct segments from the endpoints of the diameter to one of the intersections, $B$, of the perpendicular and the circle.
Step 5 Determine $A B, B C, A D, D C$, and $A C$. Use the radius to determine the exact coordinates of $B$.
Step 6 Trace the triangles onto another sheet of paper and cut apart $\Delta$ $A B D$ and $\triangle D B C$. Cut out the original $\triangle A B C$ and determine whether the three triangles are similar. Justify your answer.
2. Describe the proportional relationship including $\overline{B D}$ in $\triangle A B D$ and $\Delta$ $B C D$.

3. How can the construction in 1 be altered so that $B D$ represents the arithmetic mean?
4. What type of triangle is formed when the arithmetic mean is constructed?
5. Describe the point on the hypotenuse where the arithmetic mean is equal to the geometric mean of the lengths of the segments formed.

## Dilations

Overview: This activity invites participants to use coordinate geometry, distance, angle measure, and inductive reasoning to generalize the attributes of similar figures to formalize the definition of dilation.

Objective: TExES Mathematics Competencies
III.012.B. The beginning teacher uses properties of points, lines, planes, angles, lengths, and distances to solve problems.
III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.012.E. The beginning teacher describes and justifies geometric constructions made using compass and straightedge, reflection devices, and other appropriate technologies.
III.014.D. The beginning teacher applies transformations in the coordinate plane.
III.014.E. The beginning teacher applies concepts and properties of slope, midpoint, parallelism, perpendicularity, and distance to explore properties of geometric figures and solve problems in the coordinate plane. V.018.B. The beginning teacher applies correct mathematical reasoning to derive valid conclusions from a set of premises.
V.018.C. The beginning teacher uses inductive reasoning to make conjectures and uses deductive methods to evaluate the validity of conjectures.

## Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.D. The student uses inductive reasoning to formulate a conjecture.
d.2.A. The student uses one- and two- dimensional coordinate systems to represent points, lines, line segments, and figures.
d.2.C. The student develops and uses formulas including distance and midpoint.

Background: Participants should have some experience with translations on the coordinate plane prior to this activity.

Materials: compass, The Geometer's Sketchpad, The Geometer's Sketchpad sketch: Dilation Investigation, centimeter grid paper, straightedge

New Terms: center of dilation, contraction, dilation, expansion

## Procedures:

1. $\triangle A B C$ is drawn in the coordinate plane below. What are the coordinates of $\Delta A^{\prime} B^{\prime} C^{\prime}$ after a transformation of $(x, y) \rightarrow(2 x, 2 y)$ ? What is the relationship between $\triangle A B C$ and $\triangle A^{\prime} B^{\prime} C^{\prime}$ ?
$A(1,3) \rightarrow A^{\prime}(2,6)$
$B(3,1) \rightarrow B^{\prime}(6,2)$
$C(4,4) \rightarrow C^{\prime}(8,8)$
$\triangle A^{\prime} B^{\prime} C^{\prime}$ is similar to $\triangle A B C$ with a scale factor of 2 .

2. Draw $\Delta A^{\prime} B^{\prime} C^{\prime}$ and plot $O$ at the origin. Draw $\overrightarrow{O A}, \overrightarrow{O B}$, and $\overrightarrow{O C}$. Describe rays $\overrightarrow{O A}, \overrightarrow{O B}$, and $\overrightarrow{O C}$.
$\overrightarrow{O A}$ is a ray beginning at the origin and containing the points $A$ and $A^{\prime}$.
$\overrightarrow{O B}$ is a ray beginning at the origin and containing the points $B$ and $B^{\prime}$.
$\overrightarrow{O C}$ is a ray beginning at the origin and containing the points $C$ and $C^{\prime}$.

3. What is the relationship between $O A^{\prime}$ and $O A ; O B^{\prime}$ and $O B$, and $O C^{\prime}$ and $O C$ ?

$$
\begin{aligned}
& \frac{O A^{\prime}}{O A}=\frac{\sqrt{(2-0)^{2}+(6-0)^{2}}}{\sqrt{(1-0)^{2}+(3-0)^{2}}}=\frac{\sqrt{40}}{\sqrt{10}}=2 \\
& \frac{O B^{\prime}}{O B}=\frac{\sqrt{(6-0)^{2}+(2-0)^{2}}}{\sqrt{(3-0)^{2}+(1-0)^{2}}}=\frac{\sqrt{40}}{\sqrt{10}}=2 \\
& \frac{O C^{\prime}}{O C}=\frac{\sqrt{(8-0)^{2}+(8-0)^{2}}}{\sqrt{(4-0)^{2}+(4-0)^{2}}}=\frac{\sqrt{128}}{\sqrt{32}}=\frac{\sqrt{(4)(32)}}{\sqrt{32}}=2
\end{aligned}
$$

The distances between $O$ and the vertices of the image are twice the distance between $O$ and the corresponding vertices of the pre-image. Corresponding distances along the rays connecting the vertices of the triangles preserve the constant of proportionality and the magnification ratio or scale factor.
Discuss the center of dilation with participants.

In previous problems you found that there was a point $O$ in which the vectors connecting the vertices of the triangles intersected. This point $O$ is called a center of dilation.
4. $\triangle D E F$ is drawn below. Using only a straightedge and a compass, draw $\triangle D^{\prime} E^{\prime} F^{\prime}$ such that $\triangle D E F \sim \Delta D^{\prime} E^{\prime} F^{\prime}$ with a scale factor of 3 .
Sketches will vary depending upon the selection of the center. Participants should select a center, $O$, and draw three rays from that point through the vertices of $\Delta$
DEF. Use a compass to mark OD. Mark this distance off two times from $D$ along $\overrightarrow{O D}$ so that $O D^{\prime}$ is three times $O D$. Repeat this process for $E$ and $F$. Draw the sides of $\Delta D^{\prime} E^{\prime} F^{\prime}$.


Introduce the formal definition of a dilation to participants.
Let $O$ be a point and $k$ be a positive real number. For any point $A$, let $A^{\prime}$ be the point on $\overrightarrow{O A}$ whose distance from $O$ is $k$ times $O A$, i.e., $O A^{\prime}=k \cdot O A$. Then the correspondence $A \rightarrow A^{\prime}$ is a dilation with scale factor $k$ and center $O$.

Remind participants to add the terms center of dilation and dilation to their glossaries.
5. Using only a straightedge and a compass, draw $\Delta D^{\prime} E^{\prime} F^{\prime}$ such that
$\triangle D E F \square \triangle D^{\prime} E^{\prime} F^{\prime}$ with a scale factor of 0.5 .
Strategies will vary. One method is to find the midpoint of each segment from $O$ to the vertices of triangle $\triangle D E F$. Connect the midpoints of the segments along the translation vector to draw $\Delta D^{\prime} E^{\prime} F^{\prime}$ with side lengths one-half the length of the side lengths of $\triangle D E F$.


Explain to participants that when the scale factor $k$ is greater than one the dilation is called an expansion and when $k$ is less than one but greater than zero the dilation is called a contraction.

Remind participants to add the terms expansion and contraction to their glossaries.
6. Describe as many properties of dilations as possible.

Under dilation, collinearity, betweenness, angles, and angle measure are preserved. All pre-image segments are parallel to their image segments. Distance is not preserved.

Participants are working at the van Hiele Descriptive and Relational Levels on this activity since the Pythagorean Theorem is used to determine exact length measurements, but inductive approaches are used to make generalizations.

## Dilations

1. $\triangle A B C$ is drawn in the coordinate plane below. What are the coordinates of $\Delta A^{\prime} B^{\prime} C^{\prime}$ after a transformation of $(x, y) \rightarrow(2 x, 2 y)$ ? What is the relationship between $\triangle A B C$ and $\triangle A^{\prime} B^{\prime} C^{\prime}$ ?

2. Draw $\triangle A^{\prime} B^{\prime} C^{\prime}$ and plot $O$ at the origin. Draw $\overrightarrow{O A}, \overrightarrow{O B}$, and $\overrightarrow{O C}$. Describe rays $\overrightarrow{O A}, \overrightarrow{O B}$, and $\overrightarrow{O C}$.
3. What is the relationship between
$O A^{\prime}$ and $O A ; O B^{\prime}$ and $O B$; and $O C^{\prime}$ and $O C$ ?
4. $\triangle D E F$ is drawn below. Using only a straightedge and a compass, draw $\Delta D^{\prime} E^{\prime} F^{\prime}$ such that $\triangle D E F \sim \Delta D^{\prime} E^{\prime} F^{\prime}$ with a scale factor of 3 .

5. Using only a straightedge and a compass, draw $\Delta D^{\prime} E^{\prime} F^{\prime}$ such that $\Delta D E F \sim \Delta D^{\prime} E^{\prime} F^{\prime}$ with a scale factor of 0.5 .

6. Describe as many properties of dilations as possible.

## Similarity and the Golden Ratio

Overview: In this activity, participants examine the golden ratio as it applies to similar rectangles and other geometric figures.

## Objective: TExES Mathematics Competencies

I.001.H. The beginning teacher demonstrates how some problems that have no solution in the integer or rational number systems have solutions in the real number system.
II.004.A. The beginning teacher recognizes and extends patterns and relationships in data presented in tables, sequences and graphs. II.006.G. The beginning teacher models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology.
II.008.E. The beginning teacher models and solves problems involving exponential growth and decay.
III.011.D. The beginning teacher applies the Pythagorean theorem, proportional reasoning, and right triangle trigonometry to solve measurement problems.
III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.012.E. The beginning teacher describes and justifies geometric constructions made using compass and straightedge, reflection devices, and other appropriate technologies. III.012.F. The beginning teacher demonstrates an understanding of the use of appropriate software to explore attributes of geometric figures and to make and evaluate conjectures about geometric relationships. III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
V.018.B. The beginning teacher applies correct mathematical reasoning to derive valid conclusions from a set of premises.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.
V.019.A. The beginning teacher recognizes and uses multiple representations of a mathematical concept (e.g., a point and its coordinates, the area of a circle as a quadratic function of the radius, probability as the ratio of two areas, area of a plane region as a definite integral).
V.019.B. The beginning teacher understands how mathematics is used to model and solve problems in other disciplines (e.g., art, music, science, social science, business).
V.019.D. The beginning teacher communicates mathematical ideas using a variety of representations (e.g., numeric, verbal, graphical, pictorial, symbolic, concrete).

## Geometry TEKS

b.1.B. Through the historical development of geometric systems, the student recognizes that mathematics is developed for a variety of purposes.
b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.D. The student uses inductive reasoning to formulate a conjecture.
b.4. The student uses a variety of representations to describe geometric relationships and solve problems.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
d.2.B. The student uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons.
e.1.C. The student develops, extends, and uses the Pythagorean Theorem.
f.2. The student uses ratios to solve problems involving similar figures.

Background: Participants need a basic understanding of similarity and proportionality. They should be able to solve a proportion, a quadratic equation, and possess an understanding of exponential functions.

Materials: compass, The Geometer's Sketchpad, The Geometer's Sketchpad Sketches: Mona Lisa, Golden Construction, Spiral, graphing calculator, centimeter grid paper, patty paper, straightedge

New Terms: gnomon, golden ratio, golden rectangle, logarithmic spiral

## Procedures:

## Part A

Throughout history artists, architects, and engineers have used the golden rectangle because of its eye-pleasing proportions.

Introduce the golden ratio through an investigation of similar rectangles in the Mona Lisa. Use The Geometer's Sketchpad sketch Mona Lisa or the picture in the participant handout to determine the ratio of the length of the long side to the length of the short side of the rectangles in the picture.

The purpose of this activity is for participants to determine the similarity of the rectangles. All of the rectangles in the picture are golden, the ratio of the length of the long side to the length of the short side of the rectangles is $\Phi$. In The Geometer's Sketchpad sketch, the rectangle is constructed to remain golden throughout the investigation.

1. Measure the length of the long side and the length of the short side of the rectangles in the picture.


Musee du Louvre, Mona Lisa, Leonardo DaVinci
2. Describe the ratio of the length of the long side to the length of the short side of the rectangles.
The rectangles are different sizes but all have the same ratio of long side to short side, approximately 1.62.

Explain to participants that this ratio of the long side to the short side of the rectangles is a special number called $\Phi$ (phi, pronounced "fee"). This number not only occurs in the golden rectangle but throughout nature as well. For example, the function that
determines the logarithmic spiral shape of the chambered nautilus is based on $\Phi$. The three bones of the middle finger of a human form a $\Phi$ progression.

Remind participants to add the terms golden ratio and golden rectangle to their glossaries.

## Part B

3. Construct a golden rectangle either with The Geometer's Sketchpad or with a compass and straightedge.
Guide participants as they construct the golden rectangle using The Geometer's Sketchpad Golden Construction or compass and straightedge. A large and small rectangle will be created in the construction.

4. Are the two rectangles created in the construction similar? Explain. Yes. By measuring we can determine that the rectangles have congruent corresponding angles and proportional corresponding segments.
5. Describe the ratio of the long side to the short side in each rectangle and the scale factor that relates them.

The ratio of the long side to the short side and the scale factor are both approximately equal to 1.62. This number is an approximation of $\Phi$.
6. Use your knowledge of the construction to determine an exact value for $\Phi$, the ratio relating the sides of the rectangle. (Hint: let one side of the square be $x$.)

radius $=\sqrt{x^{2}+\left(\frac{x}{2}\right)^{2}}$
$=\sqrt{x^{2}+\frac{x^{2}}{4}}$
$=\sqrt{\frac{4 x^{2}}{4}+\frac{x^{2}}{4}}$

$$
\begin{aligned}
& A F=\frac{x}{2}+\frac{x \sqrt{5}}{2} \\
& =\frac{x+x \sqrt{5}}{2}
\end{aligned}
$$

$$
=\frac{x(1+\sqrt{5})}{2}
$$

$=\sqrt{\frac{5 x^{2}}{4}}$
$\therefore \frac{A F}{E F}=\frac{\frac{x(1+\sqrt{5})}{2}}{x}=\frac{1+\sqrt{5}}{2}=\Phi$

## Part C

Another way to determine the exact value for $\Phi$ is to solve a proportion relating the corresponding sides of two golden rectangles, one of which is formed by creating a square by folding in the short side of the original rectangle.
7. Setup and solve this proportional situation for $x$. How is your result related to the results you found in 6 ?

$$
\begin{aligned}
& \frac{1}{x-1}=\frac{x}{1} \\
& x^{2}-x=1 \\
& x^{2}-x-1=0 \\
& x=\frac{1 \pm \sqrt{1^{2}-(4)(1)(-1)}}{2} \\
& x=\frac{1+\sqrt{5}}{2}=\Phi
\end{aligned}
$$

The ratio, long side to short side, equals $\frac{x}{l}$, which is $\Phi$.

Show that $\Phi^{2}=\Phi+1$
$\left(\frac{1+\sqrt{5}}{2}\right)^{2}=\frac{1+\sqrt{5}}{2}+1$
$\frac{1+2 \sqrt{5}+5}{4}=\frac{1+\sqrt{5}}{2}+\frac{2}{2}$
$\frac{6+2 \sqrt{5}}{4}=\frac{3+\sqrt{5}}{2}$
$\frac{3+\sqrt{5}}{2}=\frac{3+\sqrt{5}}{2}$
8. Plot the graphical solution to $x^{2}=x+1$. Use your graphing calculator to find the solution in the first quadrant.

9. Draw a rectangle whose diagonal endpoints are at the origin and at the point of intersection of $y=x^{2}$ and $y=x+1$ in the first quadrant. Describe the ratio of the length of the long side to the length of the short side in this rectangle.


The ratio is $\Phi$.
10. Draw a rectangle with consecutive vertices at the origin and at point $(0,1)$ inside the original rectangle. Describe the ratio of the length of the long side to the length of the short side in this smaller rectangle. Describe the scale factor between the two rectangles.


The ratio is $\Phi$.
The scale factor is $\Phi$.
11. Draw a rectangle whose diagonal endpoints are at the origin and at the point $(1,1)$. Describe the ratio of the length of the long side to the length of the short side in this smaller rectangle. Describe the scale factor between this rectangle and the previous rectangle.


The ratio is $\Phi$.
The scale factor is $\Phi$.

## Part D

Introduce the term gnomon to participants at this time. A gnomon is a region added to a geometrical figure to make a similar larger figure. Gnomons are of great interest since many living organisms exhibit gnomonic growth. The golden spiral formed by the gnomonic growth of golden rectangles is an example of a logarithmic spiral that can be used to model the chambered nautilus.

Remind participants to add the new terms gnomon and logarithmic spiral to their glossaries.
12. Begin with the construction performed in Part B. Create subsequent rectangles by creating a square with side lengths equal to the length of the short leg of the rectangle. Repeat this process to create a golden spiral. An arc of the spiral is the arc of the circle that has a radius equal to the side of the square. See below. Use The Geometer's Sketchpad sketch Spiral.

13. Measure the long side of each golden rectangle. Create a data table beginning with the longest side. The data can also be found in The Geometer's Sketchpad sketch Spiral. Plot your data using your graphing calculator.

## Sample Data

| Term <br> Number | Side <br> Length |
| :---: | :---: |
| 1 | 6.02 |
| 2 | 3.72 |
| 3 | 2.30 |
| 4 | 1.42 |
| 5 | 0.88 |
| 6 | 0.54 |
| 7 | 0.34 |
| 8 | 0.21 |
| 9 | 0.13 |
| 10 | 0.08 |
| 11 | 0.05 |
| 12 | 0.03 |


14. Determine a function rule that models your data.

If every rectangle in the construction is related to the next larger rectangle by the scale factor $\Phi=\frac{1+\sqrt{5}}{2}$, then a larger rectangle is related to a smaller one by the scale factor $\frac{1}{\Phi}=\frac{1}{\frac{1+\sqrt{5}}{2}}=\frac{2}{1+\sqrt{5}}=\frac{2}{1+\sqrt{5}} \cdot \frac{1-\sqrt{5}}{1-\sqrt{5}}=\frac{2-2 \sqrt{5}}{-4}=\frac{-1+\sqrt{5}}{2} \approx .62$.
If this scale factor is applied repeatedly, an exponential relationship is formed. Work backwards to determine that the y-intercept is 9.74. The function is
Side length $=9.74\left(\frac{-1+\sqrt{5}}{2}\right)^{x}$ or $9.74(.62)^{x}$.
15. Test your rule by graphing over your data.


Participants are working at the van Hiele Deductive Level in this activity since deductive reasoning and justification is used to progress through the geometric and algebraic representation within the activity.

## Similarity and the Golden Ratio

## Part A

Throughout history artists, architects, and engineers have used the golden rectangle in their work because of its eye-pleasing proportions.

1. Measure the length of the long side and the length of the short side of the rectangles in the picture.


Musee du Louvre, Mona Lisa,
Leonardo DaVinci
2. Describe the ratio of the length of the long side to the length of the short side of the rectangles.

## Part B

3. Construct a golden rectangle either with The Geometer's Sketchpad or with a compass and straightedge.
4. Are the two rectangles created in the construction similar? Explain.
5. Describe the ratio of the long side to the short side in each rectangle and the scale factor that relates them.
6. Use your knowledge of the construction to determine an exact value for $\Phi$, the ratio relating the sides of the rectangle. (Hint: let one side of the square be $x$.)

## Part C

Another way to determine the exact value for $\Phi$ is to solve a proportion relating the corresponding sides of two golden rectangles, one of which is formed by creating a square by folding in the short side of the original rectangle.
7. Setup and solve this proportional situation for $x$. How is your result related to the results you found in 6 ?
8. Plot the graphical solution to $x^{2}=x+1$. Use your graphing calculator to find the solution in the first quadrant.

9. Draw a rectangle whose diagonal endpoints are at the origin and at the point of intersection of $y=x^{2}$ and $y=x+1$ in the first quadrant. Describe the ratio of the length of the long side to the length of the short side in this rectangle.

10. Draw a rectangle with consecutive vertices at the origin and at point $(0,1)$ inside the original rectangle. Describe the ratio of the length of the long side to the length of the short side in this smaller rectangle.
Describe the scale factor between the two rectangles.

11. Draw a rectangle whose diagonal endpoints are at the origin and at the point $(1,1)$. Describe the ratio of the length of the long side to the length of the short side in this smaller rectangle. Describe the scale factor between this rectangle and the previous rectangle.


## Part D

12. Begin with the construction performed in Part B. Create subsequent rectangles by creating a square with side lengths equal to the length of the short leg of the rectangle. Repeat this process to create a golden spiral. An arc of the spiral is the arc of the circle that has a radius equal to the side of the square. See below. Use The Geometer's Sketchpad sketch Spiral.

13. Measure the long side of each golden rectangle. Create a data table beginning with the longest side. The data can also be found in The Geometer's Sketchpad sketch Spiral. Plot your data using your graphing calculator.

14. Determine a function rule that models your data.
15. Test your rule by graphing the function over your data.


## Trigonometry

Overview: Participants explore trigonometric ratios and their connection to trigonometric functions.

## Objective: TExES Mathematics Competencies

II.004.A. The beginning teacher recognizes and extends patterns and relationships in data presented in tables, sequences and graphs.
II.005.B. The beginning teacher identifies the mathematical domain and range of functions and relations and determines reasonable domains for given situations.
II.005.C. The beginning teacher understands that a function represents a dependence of one quantity on another and can be represented in a variety of ways (e.g., concrete models, tables, graphs, diagrams, verbal descriptions, symbols).
II.009.A. The beginning teacher analyzes the relationships among the unit circle in the coordinate plane, circular functions, and trigonometric functions.
II.009.B. The beginning teacher recognizes and translates among various representations (e.g., written, numerical, tabular, graphical, algebraic) of trigonometric functions and their inverses. II.009.G. The beginning teacher uses graphing calculators to analyze and solve problems involving trigonometric functions.
III.011.D. The beginning teacher applies the Pythagorean Theorem, proportional reasoning, and right triangle trigonometry to solve measurement problems.
III.012.E. The beginning teacher describes and justifies geometric constructions made using compass and straightedge, reflection devices, and other appropriate technologies.
III.012.F. The beginning teacher demonstrates an understanding of the use of appropriate software to explore attributes of geometric figures and to make and evaluate conjectures about geometric relationships. V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.
V.019.A. The beginning teacher recognizes and uses multiple representations of a mathematical concept (e.g., a point and its coordinates, the area of a circle as a quadratic function of the radius, probability as the ratio of two areas, area of a plane region as a definite integral).
V.019.D. The beginning teacher communicates mathematical ideas using a variety of representations (e.g., numeric, verbal, graphical, pictorial, symbolic, concrete).

## Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.D. The student uses inductive reasoning to formulate a conjecture.
b.4. The student uses a variety of representations to describe geometric relationships and solve problems.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
c.3. The student identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-6090) and triangles whose sides are Pythagorean triples.
f.2. The student uses ratios to solve problems involving similar figures. f.3.The student develops, applies, and justifies triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples.

Background: Participants should have a basic understanding of similarity and proportionality. Participants should be able to solve a proportion, a quadratic equation, and be familiar with the Pythagorean Theorem.

Materials: cardstock, compass, The Geometer's Sketchpad, The Geometer's Sketchpad sketches: Trigonometry Ratios, Trigonometry Tracers, graphing calculator, centimeter grid paper, patty paper, protractor, scissors, straightedge

New Terms: angle in standard position, cosine, coterminal angles, initial side, periodic function, reference angle, sine, tangent, terminal side

## Procedures:

## Part A

Use The Geometer's Sketchpad sketch Trigonometry Ratios to investigate sine, cosine, and tangent. $\triangle A B C$ is a right triangle. Drag $C$. What can you say about the right triangles formed? Click on the Change Angle button to change $\angle A$, then drag $C$. What can you say about the right triangles formed?
The triangles are similar. Changing $\angle A$ produces a new set of similar triangles.


1. Click on the Show Ratios button. Drag $C$ to change the triangle. Keep $\angle A$ constant. What do you notice about the ratios?
$\frac{\text { opposite side length }}{\text { adjacent side length }}=\frac{B C}{A C}$
$\frac{\text { opposite side length }}{\text { hypotenuse length }}=\frac{B C}{A B}$
$\frac{\text { adjacent side length }}{\text { hypotenuse length }}=\frac{A C}{A B}$


The ratios remain constant as long as $\angle A$ remains constant, because all of the triangles formed by dragging $C$ are similar.
2. Change the measure of $\angle \mathrm{A}$ by clicking on the Change Angle button. Click the button to stop the change. Repeat your observations from step one.


The values are different from the previous exercise. However, the ratios remain constant because all of the triangles formed by dragging $C$ are similar as long as $\angle A$ is constant.
3. Click on the Show Tangent, Sine, Cosine button. Compare the values for the tangent, sine, and cosine (abbreviated as tan, sin, and cos respectively) to the ratios of the sides. What do you observe? Try varying the side lengths and the measure of $\angle A$. What do you observe?

| 8 7 6 5 4 4 3 2 | $m \angle A C B=90.00^{\circ}$ $m \angle C A B=29.0^{\circ}$ $m \overline{C B}=4.01 \mathrm{~cm}$ $m \overline{A C}=7.23 \mathrm{~cm}$ $m \overline{A B}=8.27 \mathrm{~cm}$ Slope $\overline{A B}=0.55$ - - - | $\begin{aligned} & \frac{m \overline{C B}}{m \overline{A C}}=0.55 \\ & \frac{m \overline{C B}}{m \overline{A B}}=0.49 \\ & \frac{m \overline{A C}}{m \overline{A B}}=0.87 \end{aligned}$ | $\begin{aligned} & \tan (m \angle C A B)=0.55 \\ & \sin (m \angle C A B)=0.49 \\ & \cos (m \angle C A B)=0.87 \end{aligned}$ | Change Angle <br> Drag $C$ to change the length of the legs of the triangle. Click on Change Angle to change the measure of $\angle A$. <br> Hide Ratios <br> Hide Tangent, Sine, Cosine |
| :---: | :---: | :---: | :---: | :---: |
| A | $2$ | 4 | $6 \quad C \quad 8$ | $10 \times 12$ |

$\tan (A)=\frac{\text { opposite side length }}{\text { adjacent side length }}=\frac{B C}{A C}$
$\sin (A)=\frac{\text { opposite side length }}{\text { hypotenuse length }}=\frac{B C}{A B}$
$\cos (A)=\frac{\text { adjacent } \text { side length }}{\text { hypotenuse length }}=\frac{A C}{A B}$
These ratios remain constant as long as $\angle A$ remains constant, because all of the triangles formed are similar.

Emphasize to participants that these are the tangent, sine, and cosine.
4. What do you notice about the slope of the hypotenuse and the tangent ratio?

The slope of the hypotenuse is equal to the tangent ratio, because the length of the opposite leg is the vertical change and length of the adjacent leg is the horizontal change.

Remind participants to add the new terms cosine, sine and tangent to their glossaries.

## Part B

Part B will relate right triangle geometry to the trigonometric functions. Trigonometric functions are based on rotations about the center of a circle. In geometry, an angle is defined as two rays with a common endpoint called a vertex. In trigonometry, an angle is defined in terms of a rotating ray. The beginning ray, called the initial side of the angle, is rotated about its endpoint. The final ray is called the terminal side of the angle. The endpoint of the ray is called the vertex of the angle (see figure below). An angle as a measure of rotation can be very large depending on the number of rotations that are made.


A clockwise and a counterclockwise angle

An angle in standard position is positioned on a rectangular coordinate system with its vertex at the origin and its initial side on the positive $x$-axis (see figure below). It is measured counter-clockwise from the horizontal axis if the angle measure is positive and clockwise from the horizontal axis if the angle measure is negative.


Two angles are coterminal if they have the same initial and same terminal sides. Coterminal angles are found by adding or subtracting an integer multiple of $360^{\circ}$. $\beta$ and $\theta$ are coterminal if and only if $\beta=\theta+n \cdot 360^{\circ}$ where $n$ is an integer (see figure above).

The reference angle of an angle in standard position is the positive, acute angle between the horizontal axis and the terminal side. Reference angles are always measured counterclockwise (see figure below).


Remind participants to add the new terms angle in standard position, coterminal angles, initial side, reference angle and terminal side to their glossaries.
5. Draw a rectangular coordinate system on a piece of easel paper. Draw a large circle centered at the origin. Let the radius of this circle be one unit (see figure below).
6. Out of cardstock cut two $30^{\circ}-60^{\circ}-90^{\circ}$ triangles and a $45^{\circ}-45^{\circ}-90^{\circ}$ triangle with the length of the hypotenuse equal to the length of the radius of the circle. Place the triangles in the first quadrant. The hypotenuse of each triangle will be the terminal side of each angle. Use your knowledge of special right triangles to determine the coordinates $(x, y)$ of the intersection of the hypotenuse and the circle for each angle measure as the terminal side of the angle is rotated around the circle. Complete the data table for the first quadrant. Reflect the triangles over the $y$-axis and complete the data table for the second quadrant. Repeat this process for the remaining quadrants and complete the data table.


| Central <br> Angle <br> $\theta$ (degrees) | $\boldsymbol{x}$ | $y$ | $r$ | $\begin{gathered} \sin \theta \\ \frac{y}{r} \end{gathered}$ | $\begin{gathered} \cos \theta \\ \frac{x}{r} \end{gathered}$ | $\begin{gathered} \tan \theta \\ \frac{y}{x} \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 30 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ |
| 45 | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 |
| 60 | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| 90 | 0 | 1 | 1 | 1 | 0 | undefined |
| 120 | $-\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 | $\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$ | $-\sqrt{3}$ |
| 135 | $-\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 | $\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | -1 |
| 150 | $-\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{3}}{3}$ |
| 180 | -1 | 0 | 1 | 0 | -1 | 0 |
| 210 | $-\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$ | 1 | $-\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ |
| 225 | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | 1 | $\frac{-}{\sqrt{2}}$ | $-\frac{\sqrt{2}}{2}$ | 1 |
| 240 | $-\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | 1 | $-\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$ | $\sqrt{3}$ |
| 270 | 0 | -1 | 1 | -1 | 0 | undefined |
| 300 | $\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | 1 | $-\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $-\sqrt{3}$ |
| 315 | $\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | 1 | - $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | -1 |
| 330 | $\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$ | 1 | $-\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{3}}{3}$ |
| 360 | 1 | 0 | 1 | 0 | 1 | 0 |

7. Enter the data from your table into your graphing calculator. Plot $\frac{y}{r}$ versus $\theta$.

8. Use your knowledge of the trigonometric ratios to write a function rule to model the data that you collected.
$y=\sin \theta$

9. What is $\sin \left(405^{\circ}\right)$ ?
$\frac{\sqrt{2}}{2}$
10. What would happen if we continued revolving the terminal side of the angle around the circle?
The angle measure would continue to increase but the values for $\sin \theta, \cos \theta$, and $\tan \theta$ would continue to repeat in the same pattern as the first revolution.
11. What would happen if we revolved the terminal side of an angle around the circle in the clockwise direction?
The patterns would repeat for negative angles.
A periodic function is a function whose values repeat in a cyclic manner. The trigonometric functions are examples of periodic functions.

Remind participants to add the term periodic function to their glossaries.
12. Describe the domains for the data set and for the function that models it.

The domain for the data set is discrete data based on the special right triangles we used. The domain for the function that models the data is all real numbers.
13. Plot $\frac{x}{r}$ versus $\theta$ on your graphing calculator.

14. Use your knowledge of the trigonometric ratios to write a function rule to model the data that you collected.
$y=\cos \theta$

15. Describe the domains for the data set and for the function that models it.

The domain for the data set is discrete data based on the special right triangles we used. The domain for the function that models the data is all real numbers.
16. Plot $\frac{y}{x}$ versus $\theta$ on your graphing calculator.

17. Use your knowledge of the trigonometric ratios to write a function rule to model the data that you collected. $y=\tan \theta$

18. Describe the domains for the data set and for the function that models it. The domain for the data set is discrete data based on the special right triangles we used. The domain for the function that models the data is all real numbers excluding $90^{\circ}+n 180^{\circ}$ where $n$ is an integer.

Participants operate at the Descriptive Level in Part 1, since they make inductive inferences from observed data. In Part 2, participants operate at the Relational Level since they correlate angle information from Part 1 with quadrant information.

Demonstrate using The Geometer's Sketchpad sketch Trigonometry Tracers.

## Trigonometry

## Part A

Use The Geometer's Sketchpad sketch Trigonometry Ratios to investigate sine, cosine, and tangent. $\triangle A B C$ is a right triangle. Drag $C$. What can you say about the right triangles formed? Click on the Change Angle button to change $\angle A$ then drag $C$. What can you say about the right triangles formed?


1. Click on the Show Ratios button. Drag $C$ to change the triangle. Keep $\angle A$ constant. What do you notice about the ratios?
2. Change the measure of $\angle A$ by clicking on the Change Angle button. Click the button to stop the change. Repeat your observations from step one.
3. Click on the Show Tangent, Sine, Cosine button. Compare the values for the tangent, sine, and cosine (abbreviated as tan, sin, and cos respectively) to the ratios of the sides. What do you observe? Try varying the side lengths and the measure of $\angle A$. What do you observe?
4. What do you notice about the slope of the hypotenuse and the tangent ratio?

## Part B

Part B will relate right triangle geometry to the trigonometric functions. Trigonometric functions are based on rotations about the center of a circle. In geometry, an angle is defined as two rays with a common endpoint called a vertex. In trigonometry, an angle is defined in terms of a rotating ray. The beginning ray, called the initial side of the angle, is rotated about its endpoint. The final ray is called the terminal side of the angle. The endpoint of the ray is called the vertex of the angle (see figure below). An angle as a measure of rotation can be very large depending on the number of rotations that are made.


A clockwise and a counterclockwise angle

An angle in standard position is positioned on a rectangular coordinate system with its vertex at the origin and its initial side on the positive $x$-axis (see figure below). It is measured counter-clockwise from the horizontal axis if the angle measure is positive and clockwise from the horizontal axis if the angle measure is negative.


Two angles are coterminal if they have the same initial side and the same terminal side. Coterminal angles are found by adding or subtracting an integer multiple of $360^{\circ}$. $\beta$ and $\theta$ are coterminal if and only if $\beta=\theta+n \cdot 360^{\circ}$ where $n$ is an integer (see figure above).

The reference angle of an angle in standard position is the positive, acute angle between the horizontal axis and the terminal side. Reference angles are always measured counter-clockwise (see figure below).

Figure 3




5. Draw a rectangular coordinate system on a piece of easel paper. Draw a large circle centered at the origin. Let the radius of this circle be one unit (see figure).
6. Out of cardstock, cut two $30^{\circ}-60^{\circ}-90^{\circ}$ triangles and a $45^{\circ}-45^{\circ}-90^{\circ}$ triangle with the length of the hypotenuse equal to the length of the radius of the circle. Place the triangles in the first quadrant. The hypotenuse of each triangle will be the terminal side of each angle. Use your knowledge of special right triangles to determine the coordinates $(x, y)$ of the intersection of the hypotenuse and the circle for each angle measure as the terminal side of the angle is rotated around the circle. Complete the data table for the first quadrant. Reflect the triangles over the $y$-axis and complete the data table for the second quadrant. Repeat this process for the remaining quadrants and complete the data table.


| Central <br> Angle <br> $\boldsymbol{\theta}$ (degrees) | $x$ | $y$ | $r$ | $\begin{gathered} \hline \sin \theta \\ \frac{y}{r} \end{gathered}$ | $\begin{gathered} \cos \theta \\ \frac{x}{r} \end{gathered}$ | $\begin{gathered} \tan \theta \\ \frac{y}{x} \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 |  |  |  |  |  |  |
| 30 |  |  |  |  |  |  |
| 45 |  |  |  |  |  |  |
| 60 |  |  |  |  |  |  |
| 90 |  |  |  |  |  |  |
| 120 |  |  |  |  |  |  |
| 135 |  |  |  |  |  |  |
| 150 |  |  |  |  |  |  |
| 180 |  |  |  |  |  |  |
| 210 |  |  |  |  |  |  |
| 225 |  |  |  |  |  |  |
| 240 |  |  |  |  |  |  |
| 270 |  |  |  |  |  |  |
| 300 |  |  |  |  |  |  |
| 315 |  |  |  |  |  |  |
| 330 |  |  |  |  |  |  |
| 360 |  |  |  |  |  |  |

7. Enter the data from the table into your graphing calculator. Plot $\frac{y}{r}$ versus $\theta$.
8. Use your knowledge of the trigonometric ratios to write a function rule to model the data that you collected.
9. What is $\sin \left(405^{\circ}\right)$ ?
10. What would happen if we continued revolving the terminal side of an angle around the circle?
11. What would happen if we revolved the terminal side of an angle around the circle in the clockwise direction?
12. Describe the domains for the data set and for the function that models it.
13. Plot $\frac{x}{r}$ versus $\theta$ on your graphing calculator.
14. Use your knowledge of the trigonometric ratios to write a function rule to model the data that you collected.
15. Describe the domains for the data set and for the function that models it.
16. Plot $\frac{y}{x}$ versus $\theta$ on your graphing calculator.
17. Use your knowledge of the trigonometric ratios to write a function rule to model the data that you collected.
18. Describe the domains for the data set and for the function that models it.

## Exploring Pyramids and Cones

Overview: $\quad$ Participants explore the volumes of pyramids through a wire-frame construction. Participants also extend their knowledge of similarity to comparing the surface areas of similar cones.

## Objective: TExES Mathematics Competencies

II.004.A. The beginning teacher recognizes and extends patterns and relationships in data presented in tables, sequences, or graphs. II.006.G. The beginning teacher models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology.
III.011.B. The beginning teacher applies formulas for perimeter, area, surface area, and volume of geometric figures and shapes (e.g., polygons, pyramids, prisms, cylinders, cones, spheres) to solve problems.
III.011.C. The beginning teacher recognizes the effects on length, area, or volume when the linear dimensions of plane figures or solids are changed.
III.012.E. The beginning teacher describes and justifies geometric constructions made using compass and straightedge, reflection devices, and other appropriate technologies.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
III.013.E. The beginning teacher analyzes cross-sections and nets of three-dimensional shapes.
V.018.C. The beginning teacher uses inductive reasoning to make conjectures and uses deductive methods to evaluate the validity of conjectures.
V.019.A. The beginning teacher recognizes and uses multiple representations of a mathematical concept (e.g., a point and its coordinates, the area of a circle as a quadratic function of the radius, probability as the ratio of two areas, area of a plane region as a definite integral).

## Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
b.3.D. The student uses inductive reasoning to formulate a conjecture.
b.4. The student uses a variety of representations to describe geometric relationships and solve problems.
d.1.A. The student describes, and draws cross sections and other slices of three-dimensional objects.
d.1.B. The student uses nets to represent and construct threedimensional objects.
d.2.C. The student develops and uses formulas including distance and midpoint.
e.1.D. The student finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations. e.2.D. The student analyzes the characteristics of three-dimensional figures and their component parts.
f.1. The student uses similarity properties and transformations to explore and justify conjectures about geometric figures.
f.4. The student describes the effect on perimeter, area, and volume when length, width, or height of a three-dimensional solid is changed and applies this idea in solving problems.

Background: Participants should recall how to find area, surface area, and volume from the unit on area.

Materials: wire-frame constructions from Exploring Prisms, centimeter rulers, scissors, protractor, patty (tracing) paper (optional), compass (optional), paper cone-shaped drinking cups, plastic rice, cardstock

New Terms: cone, oblique pyramid, pyramid, right cone, right pyramid, slant height of a cone

## Procedures:

1. Using your wire-frame prism from Exploring Prisms, remove the polygon that forms the top base. Use a small piece of modeling clay to bring the free ends of the floral wire pieces together to one point. Sketch what you see below. Label the critical attributes.
Answers will vary depending on which polygon each group constructed. An example is shown below. Define a pyramid as a polyhedron with three or more triangular faces that meet at a point and one other face, a polygon, which is called the base (Geometry To Go, 2001). Discuss attributes such as altitude, edge, face, vertex, height, slant height, edge length, base, and area of base.


Using the regular polygon you removed from your wire-frame prism, construct the angle bisectors using a protractor, compass, or a piece of patty paper. If the base polygon is a rectangle, draw the diagonals. Using a pair of scissors, carefully cut along each angle bisector from the vertex stopping short of the point of intersection.
It is important that you do not cut all the way to the point of intersection. Carefully insert this polygon inside the wire-frame pyramid or cone you have constructed. One piece of wire should fit inside each angle bisector slit.
2. Describe what happens when the polygon slides up and down the pieces of floral wire.
As the polygon moves up and down (out of the plane of the base), the pieces of floral wire create vertices of a similar polygon that is dilated by a certain scale factor.
Theoretically, the polygon would continue dilating until the vertices all meet at the apex of the pyramid.


When the apex does not lie on the line containing the altitude of the pyramid or cone, the solid is called oblique. When the apex lies on the line containing the altitude of the pyramid or cone, the solid is called a right pyramid or right cone since the height and the base of the solid are perpendicular, forming a right angle.
3.Recall your earlier work with prisms. What do the pieces of floral wire represent? The floral wire pieces represent the dilation vectors coming out of the plane of the base.

4. What does the piece of modeling clay at the top of your wire-frame pyramid represent?
The modeling clay represents the apex of the pyramid. It also represents the vanishing point, or center of dilation, where the dilation vectors intersect. Geometrically, this point is called the "center of dilation." Artists use the term "vanishing point," since objects become proportionally smaller as they approach this point until they scale to zero or vanish.

Define a cone as a three-dimensional solid which
"like a pyramid... has a base and a vertex. The base of a cone is a circle and its interior. The radius of a cone is the radius of the base. The vertex of a cone is a point not in the same plane as the base. The altitude of a cone is the perpendicular segment from the vertex to the plane of the base. The height of a cone is the length of the altitude." (Serra, 2003, p. 533)

Take a paper cone-shaped drinking cup and measure the diameter of the base of the cone and the height of the cone. Record your measurements. Construct a cylinder with the same dimensions out of cardstock. Fill the paper coneshaped cup with plastic rice.
5.Predict how many paper cones full of plastic rice it will take to exactly fill the cylinder you constructed.
Answers will vary.
6.Pour the plastic rice from the cone into the cylinder. Continue pouring cones of plastic rice into the cylinder, carefully counting how many cones full of plastic rice it takes to fill the cylinder completely. Compare the results to your prediction. Explain any similarities or differences.
It should take three cones to fill the cylinder. Explanations will vary.
7. Based on your earlier experiences with volume, what formula would you use to find the volume of a cone? Explain your reasoning. Participants should reason that since it took three cones to fill the cylinder, the volume of a cone is $\frac{1}{3}$ the volume of a cylinder with the same base and altitude. Since the formula for the volume of a cylinder is $V=\pi r^{2} h$, the formula for the volume of a cone should be $V=\frac{1}{3} \pi r^{2} h$.
Distribute the Nets for Cones activity sheet. Review the concept of nets as a two-dimensional representation of a three-dimensional solid that can be cut out and folded to construct the solid object.
8. Use a ruler to measure the slant height and radius of each cone to the nearest tenth of a centimeter.
Define "slant height of a right cone" as the length of a segment from the apex to a point on the circumference of the base.
9. Construct the three cones from their nets.
10. What methods can you use to find the surface area of a cone?

Answers may vary. Some participants may use formulas.
To develop the formula for the surface area of a cone, ask participants to consider each piece of the net of the cone.

## What shapes are the two pieces?

The base of the cone is a circle with a radius, $r$, and the lateral surface is a sector of a circle with a radius, $l$, the slant height of the cone.

## How do you find the area of each of these pieces?

The area of the circle can be found with the formula $A=\pi r^{2}$. The area of a sector of a circle can be found by considering the fractional part of the area of the circle that the area of the sector represents. Recall, the fractional part of the circle, which the sector represents, can be obtained using the ratio of the central angle of the sector to $360^{\circ}$, which is equal to the ratio of the arc length of the sector to the circumference length of the circle, with the same radius as the sector, in this case, $l$.

Notice that the arc length of the sector is equal to the length of the circumference of the circle representing the base of the cone, i.e., the arc length of the sector $=2 \pi r$.
The fractional part of the area of the circle which the area of the sector represents equals $\frac{2 \pi r}{2 \pi l}$.

The formula for the surface area of a cone is developed as follows.

$$
\begin{aligned}
& \text { Total Surface Area }=\text { Area of sector }+ \text { Area of circle } \\
& \qquad \begin{aligned}
S A & =\frac{2 \pi r}{2 \pi l}\left(\pi l^{2}\right)+\pi r^{2} \\
S A & =\pi r l+\pi r^{2}
\end{aligned}
\end{aligned}
$$

11. Find the surface area of each cone.

Large cone: $95.5 \mathrm{~cm}^{2}$
Medium cone: $53.7 \mathrm{~cm}^{2}$
Small cone: $23.6 \mathrm{~cm}^{2}$
12. Use the cones that you constructed to complete the table below.

| Cone | Radius <br> of Base <br> in cm | Slant <br> Height of <br> Cone in | Surface <br> Area of <br> Cone in <br> $\mathbf{c m}^{2}$ | Ratio of <br> Radii | Ratio of Surface <br> Areas |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Large | 2 | 13.2 | 95.5 | $\frac{r_{1}}{r_{1}}=\frac{1}{l}$ | $\frac{S A_{1}}{S A_{1}}=\frac{95.5}{95.5}=\frac{1}{1}$ |
| Medium | 1.5 | 9.9 | 53.7 | $\frac{r_{1}}{r_{2}}=\frac{2}{1.5}=\frac{4}{3}$ | $\frac{S A_{1}}{S A_{2}}=\frac{95.5}{53.7} \approx \frac{16}{9}$ |
| Small | 1 | 6.5 | 23.6 | $\frac{r_{1}}{r_{3}}=\frac{2}{1}$ | $\frac{S A_{1}}{S A_{3}}=\frac{95.5}{23.6} \approx \frac{4}{1}$ |

13. What patterns do you notice in the ratios above?

Answers may vary but should include that the ratios of the surface areas are the squares of the ratios of the radii.

Remind participants to add the new terms cone, oblique pyramid, pyramid, right cone, right pyramid, and slant height of a cone to their glossaries.

Success in this activity indicates that participants are working at the van Hiele Relational Level because they must discover the relationship between the arc length of the sector, the circumference of the circle representing the base of the cone and the fraction of the circle which the sector represents and make simple deductions to determine the total surface area of the cone.


## Exploring Pyramids and Cones

1. Using your wire-frame prism from Exploring Prisms, remove the polygon that forms the top base. Use a small piece of modeling clay to bring the free ends of the floral wire pieces together to one point. Sketch what you see below. Label the critical attributes.

Using the polygon you removed from your wire-frame prism, construct the angle bisectors using a protractor, compass, or a piece of patty paper. Using a pair of scissors, carefully cut along each angle bisector from the vertex stopping short of the point of intersection. It is important that you do not cut all the way to the point of intersection. Carefully insert this polygon inside the wire-frame pyramid or cone you have constructed. One piece of wire should fit inside each angle bisector slit.
2. Describe what happens when the polygon slides up and down the pieces of floral wire.

When the apex does not lie on the line containing the altitude of the pyramid or cone, the solid is called oblique. When the apex lies on the line containing the altitude of the pyramid or cone, the solid is called a right pyramid or right cone since the height and the base of the solid are perpendicular, forming a right angle.
3. Recall your earlier work with prisms. What do the pieces of floral wire represent?
4. What does the piece of modeling clay at the top of your wire-frame pyramid represent?

Take a paper cone-shaped drinking cup and measure the diameter of the base of the cone and the height of the cone. Record your measurements. Construct a cylinder with the same dimensions out of cardstock. Fill the paper cone-shaped cup with plastic rice.
5. Predict how many paper cones full of plastic rice it will take to exactly fill the cylinder you constructed.
6. Pour the plastic rice from the cone into the cylinder. Continue pouring cones of plastic rice into the cylinder, carefully counting how many cones full of plastic rice it takes to fill the cylinder completely. Compare the results to your prediction. Explain any similarities or differences.
7. Based on your earlier experiences with volume, what formula would you use to find the volume of a cone? Explain your reasoning.
8. Use a ruler to measure the slant height and radius of each cone to the nearest tenth of a centimeter.
9. Construct the three cones from their nets.
10. What methods can you use to find the surface area of a cone?
11. Find the surface area of each cone.
12. Use the cones that you constructed to complete the table below.

| Cone | Radius of <br> Base in <br> cm | Slant <br> Cone in <br> cm | Surface <br> Area of <br> Cone in <br> $\mathbf{c m}^{2}$ | Ratio of <br> Radii | Ratio of <br> Surface <br> Areas |
| :---: | :---: | :---: | :---: | :---: | :--- |
| Large |  |  |  | $\frac{r_{1}}{r_{1}}=$ | $\frac{S A_{1}}{S A_{1}}=$ |
| Medium |  |  |  | $\frac{r_{1}}{r_{2}}=$ | $\frac{S A_{1}}{S A_{2}}=$ |
| Small |  |  |  | $\frac{r_{1}}{r_{3}}=$ | $\frac{S A_{1}}{S A_{3}}=$ |

13. What patterns do you notice in the ratios above?

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# Unit 9 - Non-Euclidean Geometries 

## When Is the Sum of the Measures of the Angles of a Triangle Equal to $\mathbf{1 8 0}^{\circ}$ ?

Overview: $\quad$ This activity illustrates the need for Euclid's Fifth Postulate in proving that the sum of the measures of the angles of a triangle is $180^{\circ}$ in Euclidean space. The negation of this theorem leads to other geometries.

Objective: TExES Mathematics Competencies
III.012.A. The beginning teacher understands axiomatic systems and their components (e.g., undefined terms, defined terms, theorems, examples, counterexamples).
III.012.E. The beginning teacher describes and justifies geometric constructions made using compass and straightedge, reflection devices, and other appropriate technologies.
III.012.G. The beginning teacher compares and contrasts the axioms of Euclidean geometry with those of non-Euclidean geometry (i.e., hyperbolic and elliptic geometry).
V.018.A. The beginning teacher understands the nature of proof, including indirect proof, in mathematics.
V.018.B. The beginning teacher applies correct mathematical reasoning to derive valid conclusion from a set of premises. V.018.C. The beginning teacher uses inductive reasoning to make conjectures and uses deductive methods to evaluation the validity of conjectures.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.
V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

## Geometry TEKS

b.1.A. The student develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems.
b.1.C. The student compares and contrasts the structures and implications of Euclidean and non-Euclidean geometries.
b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.C. The student demonstrates what it means to prove mathematically that statements are true.
b.3.E. The student uses deductive reasoning to prove a statement.

| Background: | Participants should have some knowledge of the structure of a <br> mathematical system, connecting definitions, postulates, logical <br> reasoning, and theorems as well as the properties of parallel lines in <br> Euclidean space. |
| :--- | :--- |
| Materials: | straightedge, compass, patty paper, colored pencils, transparency <br> sheet, scissors, overhead projector pens |

## New Terms:

## Procedures:

This introductory activity motivates the discussion of different geometries. The theorem that participants are asked to prove gives rise to the question "Are there geometries for which this theorem does not hold?" Historically, mathematicians tried to prove Euclid's parallel postulate (stated below in two forms). Their failure to prove it gave rise to hyperbolic and elliptic geometries. In elliptic geometry, the sum of the measures of the angles of a triangle is greater than $180^{\circ}$. In hyperbolic geometry, the sum of the measures of the angles of a triangle is less than $180^{\circ}$.

Euclid's fifth postulate (the parallel postulate) is usually stated as follows: "Through a point not on a line, there exists exactly one line parallel to the line." This version of the parallel postulate is known as Playfair's postulate (1795). It is logically equivalent to Euclid's original fifth postulate which states that "if a transversal intersects two lines so that the sum of the measures of the interior angles on the same side of the transversal is less that $180^{\circ}$, then the two lines will intersect on the side of the transversal where the interior angles are formed." You may want to have participants illustrate Euclid's original fifth postulate to see that it is logically equivalent to Playfair's postulate.

Before participants prove that the sum of the measures of the angles of a triangle is equal to $180^{\circ}$, you may want to demonstrate a "proof" of the theorem visually at the overhead projector by following these steps:

1. Draw any triangle on a piece of transparency paper.
2. Label each angle in the interior of the triangle close to each vertex.
3. Carefully tear or cut off each angle (tear or cut off each angle so that the result is three sector-like shaped regions).
4. Arrange the three angles so that their vertices meet at the same point. A straight angle is formed whose measure is $180^{\circ}$.
5. Therefore the sum of the measures of the three angles is $180^{\circ}$.


The above illustration is taken from Discovering Geometry: An Inductive Approach: $3^{\text {rd }}$ Edition, © 2003, p 199 with permission from Key Curriculum Press.

Success with this activity indicates that participants are working at the Relational Level, because they are able to produce an informal argument using a diagram and concrete materials.

1. Prove: The sum of the measures of the angles of a triangle is $180^{\circ}$.

Use the following hints to construct your proof:
(1) Begin by carefully drawing any triangle, $\triangle A B C$.
(2) Construct line $l$ through one of the vertices $C$ parallel to the opposite side $\overline{A B}$ of the triangle.
(3) Use your knowledge of angle relationships for parallel lines cut by a transversal and the fact that the measure of a straight angle is $180^{\circ}$ to prove the theorem.

Participants may construct a line, $l$, parallel to $\overline{A B}$ using a compass and straightedge or by folding patty paper.

The proof appears below:


Given: $\triangle A B C$

Prove: $m \angle A+m \angle 2+m \angle B=180^{\circ}$

Proof: $\quad \triangle A B C$
Through C, construct
line $l$ parallel to $\overline{A B}$.
(Given)
(Through a point not on a line, there exists exactly one line parallel to the line.)

$$
\begin{array}{ll}
\angle 1 \cong \angle A \text { and } \angle 3 \cong \angle B & \begin{array}{l}
\text { (If two parallel lines are cut by a } \\
\text { transversal, then the alternate interior } \\
\text { angles are congruent.) }
\end{array} \\
m \angle 1+m \angle 2+m \angle 3=180^{\circ} & \text { (A straight angle measures } 180^{\circ} . \text { ) } \\
m \angle A+m \angle 2+m \angle B=180^{\circ} & \text { (Substitution) }
\end{array}
$$

Therefore, the sum of the measures of the angles of a triangle is $180^{\circ}$.
Success with this activity indicates that participants are working at the Deductive Level, because they construct a formal proof.
2. Is this theorem always true? Explain your answer.

It is true in Euclidean geometry in which Euclid's fifth postulate holds. It is not true for geometries for which Euclid's fifth postulate does not hold.

Participants should recognize that without the parallel postulate, it would not be possible to prove that the sum of the measures of the angles of a triangle is $180^{\circ}$. In fact, this statement is also equivalent to Euclid's fifth postulate. You may want to ask participants to verify this equivalence.

## When Is the Sum of the Measures of the Angles of a Triangle Equal to $\mathbf{1 8 0}^{\circ}$ ?

1. Prove: The sum of the measures of the angles of a triangle is $180^{\circ}$.

Use the following hints to construct your proof:
(1) Begin by carefully drawing any triangle, $\triangle A B C$.
(2) Construct line $l$ through one of the vertices, $C$, parallel to the opposite side $\overline{A B}$ of the triangle.
(3) Use your knowledge of angle relationships for parallel lines cut by a transversal and the fact that the measure of a straight angle is $180^{\circ}$ to prove the theorem.
2. Is this theorem always true? Explain your answer.

## Euclid's First Five Postulates in Euclidean Space

Overview: In this activity, participants review the parallel postulate as well as Euclid's first four postulates in Euclidean space.

Objective: TExES Mathematics Competencies<br>III.012.A. The beginning teacher understands axiomatic systems and their components (e.g., undefined terms, defined terms, theorems, examples, counterexamples).<br>III.012.G. The beginning teacher compares and contrasts the axioms of Euclidean geometry with those of non-Euclidean geometry (i.e., hyperbolic and elliptic geometry).<br>V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.<br>V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

## Geometry TEKS

b.1.A. The student develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems.
b.1.C. The student compares and contrasts the structures and implications of Euclidean and non-Euclidean geometries. b.3.C. The student demonstrates what it means to prove mathematically that statements are true.

Background: Participants should have prior knowledge of Euclid's first five postulates in Euclidean space and should be familiar with visual representations for Euclidean space.

Materials: straightedge, protractor

## New Terms:

## Procedures:

Before participants explore Euclid's five postulates in other geometries, they should review the postulates in the familiar Euclidean space. Remind participants that postulates, or axioms, are truths that are accepted without proof. Early mathematicians tried to deduce Euclid's fifth postulate from the other four postulates because of its perceived complexity with respect to the other four.

Using the activity sheet, have participants, in groups, review Euclid's first five postulates. They should be able to illustrate the five postulates in Euclidean space.

Euclid's five postulates are:

1. For any two distinct points, there is exactly one line that contains them.
2. Any segment may be extended indefinitely in a straight line.
3. Given a point (center) and a distance (radius), a circle can be drawn.
4. All right angles are congruent.
5. Through a point not on a line, there exists exactly one line parallel to the line (Playfair's postulate).

Participants may not remember that the above five statements are Euclid's first five postulates, but after reading them, they should be able to illustrate them without difficulty.

You may want to review the process of writing negations of a given statement before participants complete 6.

You may use the example "All women love mathematics" to review negations. Possible negations of this statement are "it is false that all women love mathematics", "it is not the case that all women love mathematics", "not all women love mathematics" or "some women do not love mathematics."

If you feel participants need more practice in writing negations to given statements, have them, individually, create a couple of statements. Then have pairs or groups of three participants write all the possible negations for all the statements in their group.
6. One negation of Euclid's fifth postulate is "Through a point not on a line, there exists no line parallel to the line." State another negation for Euclid's fifth postulate. Through a point not on a line, there exists more than one line parallel to the line.

These two negations of Euclid's fifth postulate led to two non-Euclidean geometries. Elliptic geometry resulted from the first negation and hyperbolic from the second negation).

Success with 1-6 indicates that participants are working at the Deductive Level, because they are asked to demonstrate an understanding of postulates and to examine the effects of changing a postulate.

## Euclid's First Five Postulates in Euclidean Space

1-5 are statements of Euclid's first five postulates. Illustrate each of the postulates in the spaces provided below which represent Euclidean space.

1. For any two distinct points, there is exactly one line that contains them.
2. Any segment may be extended indefinitely in a straight line.
3. Given a point (center) and a distance (radius), a circle can be drawn.
4. All right angles are congruent.
5. Through a point not on a line, there exists exactly one line parallel to the line (Playfair's postulate).
6. One negation of Euclid's fifth postulate is "Through a point not on a line, there exists no line parallel to the line." State another negation for Euclid's fifth postulate.

## Curvature in Different Geometries

Overview: In this activity, participants compare and contrast the curvature of different geometries (Euclidean, elliptic, and hyperbolic).

## Objective: TExES Mathematics Competencies

III.012.A. The beginning teacher understands axiomatic systems and their components (e.g., undefined terms, defined terms, theorems, examples, counterexamples).
III.012.G. The beginning teacher compares and contrasts the axioms of Euclidean geometry with those of non-Euclidean geometry (i.e., hyperbolic and elliptic geometry).
V.018.C. The beginning teacher uses inductive reasoning to make conjectures and uses deductive methods to evaluate the validity of conjectures.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

## Geometry TEKS

b.1.A. The student develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems.
b.1.C. The student compares and contrasts the structures and implications of Euclidean and non-Euclidean geometries.
b.3.C. The student demonstrates what it means to prove mathematically that statements are true.
b.3.D. The students uses inductive reasoning to formulate a conjecture.

Background: Participants should have some knowledge of a variety of surfaces.

## Materials:

New Terms: Gaussian curvature

## Procedures:

The Gaussian curvature of a surface characterizes the geometry of the surface. It describes the intrinsic geometry of the surface and does not change even if the surface is bent without stretching or compressing it. The Euclidean plane (a flat surface) has zero Gaussian curvature. An elliptic surface (a ball or a sphere) has positive Gaussian curvature. A hyperbolic surface (the seat of a saddle) has negative Gaussian curvature. The closer to zero the Gaussian curvature is, the flatter the surface. The further from zero the Gaussian curvature is, the more sharply curved the surface, either negatively (hyperbolic) or positively (elliptic). Remind participants to add the new term Gaussian curvature to their glossaries.

The sphere is a surface with constant Gaussian curvature that is positive, and the pseudosphere has constant Gaussian curvature that is negative. The illustrations below are of surfaces with constant Gaussian curvature.


Sphere (positive)


Pseudosphere (negative)


Euclidean plane (zero)

The Gaussian curvature of a surface is not always constant as in the case of the hyperboloid paraboloid or the ellipsoid. The illustrations below are of surfaces that do not have constant Gaussian curvature.

hyperbolic paraboloid

ellipsoid

Select a point on a surface of positive Gaussian curvature such as a sphere or ellipsoid. The surface lies completely to one side of a tangent plane through that point and touches the tangent plane at exactly one point. Select a point on a surface of negative Gaussian curvature. The tangent plane through that point passes through the surface. For a surface of zero Gaussian curvature, the tangent plane contains a line that touches the surface at all the points on that line.

1. Compare the Gaussian curvature of a flat surface to the sum of the angles of a triangle on that surface.
A surface with zero Gaussian curvature is flat. The sum of the measures of the angles of a triangle on its surface is $180^{\circ}$.
2. Describe the Gaussian curvature of a flat surface. Does Euclid's parallel postulate hold for that surface?
A surface with zero Gaussian curvature is flat. The sum of the measures of the angles of a triangle on that surface is $180^{\circ}$. This statement is logically equivalent to Euclid's parallel postulate. Therefore, the parallel postulate holds for a flat surface.
3. Explain why a cylindrical surface (a cylinder without a top or a bottom) has zero Gaussian curvature.
The Gaussian curvature of a flat surface is zero. Since the lateral area of a cylinder is a flat surface, the Gaussian curvature is the same for both surfaces. Remember that bending (or in this case rolling) a surface without stretching or compressing it does not change its Gaussian curvature. Hence the Gaussian curvature of a cylinder must be zero.

The illustration below shows a cylinder and the flat surface that was rolled to form the cylinder. Notice that the tangent plane to the cylinder contains a straight line that touches the cylinder at all points on that line. This supports the claim that the Gaussian curvature of a cylinder is zero.


Participants work at the highest van Hiele level, Rigor, as they are asked to work in a variety of axiomatic systems and to consider formal abstract aspects of deduction.

For further readings on Gaussian curvature, you may want to consult Baragar (2001), Greenberg (1993), Thurston (1997), or Weeks (1985).

## Various Surfaces

## Surfaces of Constant Gaussian Curvature



Sphere (positive)
Pseudosphere (negative)
Euclidean plane (zero)

## Surfaces without Constant Gaussian Curvature


hyperbolic paraboloid


## Curvature in Different Geometries

The Gaussian curvature of a surface characterizes the geometry of a surface. It describes the intrinsic geometry of the surface. A flat (Euclidean) plane has zero Gaussian curvature.

1. Compare the Gaussian curvature of a flat surface to the sum of the angles of a triangle on that surface.
2. Describe the Gaussian curvature of a flat surface. Does Euclid's parallel postulate hold for that surface?
3. Explain why a cylindrical surface (a cylinder without a top or a bottom) has zero Gaussian curvature.

## Euclid's First Five Postulates in Elliptic Space

Overview: In this activity, participants explore Euclid's first five postulates in elliptic space.

## Objective: TExES Mathematics Competencies

III.012.A. The beginning teacher understands axiomatic systems and their components (e.g., undefined terms, defined terms, theorems, examples, counterexamples).
III.012.G. The beginning teacher compares and contrasts the axioms of Euclidean geometry with those of non-Euclidean geometry (i.e., hyperbolic and elliptic geometry).
V.018.C. The beginning teacher uses inductive reasoning to make conjectures and uses deductive methods to evaluation the validity of conjectures.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.
V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

## Geometry TEKS

b.1.A. The student develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems.
b.1.C. The student compares and contrasts the structures and implications of Euclidean and non-Euclidean geometries. b.3.C. The student demonstrates what it means to prove mathematically that statements are true.
b.3.D. The students uses inductive reasoning to formulate a conjecture.

Background: Participants should have prior knowledge of Euclid's first five postulates for Euclidean space.

Materials: flexible protractor, string, overhead projector pens, globe, beach ball, or Lénárt sphere

New Terms: elliptic geometry or elliptic space, geodesic, great circle

## Procedures:

Before participants examine Euclid's first five postulates in elliptic space, you may want to review the undefined term line for Euclidean space. This term and others like it (point, plane, and space) do not have nor do they need definitions. Recall, that in an axiomatic system, there are undefined terms, defined terms, postulates, theorems, and rules of logic.

If necessary, have participants review Euclid's first five postulates from the previous activity, carefully noting the given negation of Euclid's fifth postulate: "Through a point not on a line, there exists no line parallel to the line."

Elliptic geometry, which is also referred to as spherical geometry, is a geometry on a curved surface such as a sphere, globe, ball, egg, or an ellipsoid. The illustrations below are examples of elliptic space.


Sphere

ellipsoid

You may want to introduce the term geodesic. A geodesic is a curve that minimizes the distance between two points. In Euclidean space, geodesics are straight lines. The discussion of geodesics for elliptic space, in general, is beyond the scope of this unit. As a result, we will restrict the discussion to the sphere, the elliptic surface of constant positive Gaussian curvature.

1. What is the shortest distance between two points on the sphere?

Hint: Locate two points on the surface of your globe, beach ball, or Lénárt sphere.
Using string, find the shortest distance between your two points. The shortest distance is an arc. Trace this arc. This arc lies on a great circle which is the set of points on the surface formed by the intersection of the sphere and a plane passing through the center of the sphere. Extend your arc to find the great circle that contains it. The shortest distance between two points on the sphere lies along the great circle that contains the two points. Therefore geodesics on the sphere are great circles. Note: The shortest distance between two points on a sphere is found on the surface and never in the interior, as the interior is not part of the spherical surface.

## What are the "lines" in elliptic geometry?

On a sphere, the geodesics are great circles. A great circle is the set of points on the sphere formed by the intersection of the sphere and a plane passing through the center of the sphere. If we consider a globe, the equator and the meridians which pass through the two poles are examples of great circles. Notice that "latitude lines" are not "lines" on the sphere, except for the equator, because the plane containing them does not pass through the center of the sphere. The north and south poles are diametrically opposite each other on the surface of the earth. They are often referred to as polar points. Any pair of points that are diametrically opposite each other on the sphere may also be referred to as polar points.

Great circles divide the sphere into two congruent parts. The equator of the earth in the illustration below is a great circle. The equator divides the earth into the north and the south hemispheres.


You may want participants to informally explore and conjecture what the geodesics on an ellipsoid might be.
2. Is there more than one great circle passing through two points on your sphere?

Explain.
There is only one great circle that passes through two points unless the two points are diametrically opposite each other on a given diameter (polar points).
3. Find several examples of great circles on your sphere. Are great circles infinite in length? Why or why not?
Great circles never end although they retrace themselves. Therefore, they are finite in length. Since all the great circles on your model have the same diameter, they are all the same length.
4. Can great circles be parallel? Explain.

No, great circles can never be parallel, because any two great circles intersect. Therefore, there are no parallel lines on the sphere.
5. Locate three non-collinear points on your sphere. To form an elliptic triangle through your three points, draw the three great circles that connect pairs of these points. Measure each angle of your elliptic triangle. What is the sum of the measures of the three angles? Can you find a triangle whose three angles add up to $270^{\circ}$ ? Can you find a triangle whose three angles add up to $360^{\circ}$ ? Can you find a triangle whose three angles add up to more than $360^{\circ}$ ?
Answers will vary as the sum of the measures of the angles of an elliptic triangle depends upon the size of the triangle. However, the sum is always greater than $180^{\circ}$. A small elliptic triangle seems almost flat so that the sum of its three angles is close to $180^{\circ}$. The sum of the angles of an elliptic triangle with a right angle at the North Pole and the other two vertices on the equator is $270^{\circ}$. Therefore, two great circles perpendicular to the same line (the equator) are not parallel but meet at the North Pole. By increasing the size of the angle at the North Pole to $180^{\circ}$ or larger, you can find triangles whose angles add up to $360^{\circ}$ or more.

The illustration below is of an elliptic triangle on the sphere. It is evident that the sum of the angles of the triangle is greater than $180^{\circ}$ because the surface "bulges" out.

6. Can you find similar triangles on your sphere that are not congruent? Explain. No, similar triangles must be congruent.
7. Restate the theorem "The sum of the measures of the angles of a triangle is $180^{\circ}$ " so that it applies to elliptic space.
The sum of the measures of the angles of an elliptic triangle is always greater than $180^{\circ}$.
8. Is there a relationship between the size of a triangle on your sphere and the sum of the measures of its angles? Explain.
The larger the triangle, the greater the sum of its angles is. The smaller the triangle, the closer the sum of the angles is to $180^{\circ}$. Small elliptic triangles seem to resemble Euclidean triangles, because they are almost flat. Large elliptic triangles are more curved so that the sum of the measures of their angles is much larger than $180^{\circ}$.
9. Restate Euclid's first five postulates so that they apply to the sphere.

- For any two distinct points, there may be one or an infinite number of great circles that contain(s) them. It depends on the location of the points. If they are diametrically opposite each other, then there are an infinite number of great circles that contain them. If they are not diametrically opposite each other, then there is exactly one great circle that contains them.
- Any arc may be extended indefinitely on a great circle. However, great circles are not infinite in length.

Recall that great circles never end although they retrace themselves.

- Given a point (center) and a distance (radius), a circle can be drawn. [Unchanged from Euclidean space]

The largest circle on the sphere is a great circle.

- All right angles are congruent. [Unchanged from Euclidean space]
- Through a point not on a great circle, no great circle is parallel to it. Any two great circles intersect.

10. Compare the Gaussian curvature of an elliptic surface to the sum of the angles of a triangle on its surface.
A surface with positive Gaussian curvature is elliptic. The sum of the measures of the angles of a triangle on its surface is greater than $180^{\circ}$.
11. Describe the Gaussian curvature of an elliptic surface. Does Euclid's parallel postulate hold for its surface?
A surface with a positive Gaussian curvature is elliptic. The sum of the measures of the angles of a triangle on its surface is greater than $180^{\circ}$. The parallel postulate does not hold on that surface. There are no parallel lines in elliptic space.

For further investigations of elliptic space, you may want to refer to Rice University Mathematics Professor John Polking's web site at http://math.rice.edu/ pcmi/sphere/

Remind participants to add the new terms geodesic, elliptic space, and great circle to their glossaries.

## Euclid's First Five Postulates in Elliptic Space

You will need a globe, a beach ball, or a Lénárt sphere, tape measure or flexible ruler, flexible protractor, overhead projector pens, string, and scissors for this activity. Refer to the previous activity sheet for Euclid's first five postulates.

1. What is the shortest distance between two points on the sphere?

Hint: Locate two points on the surface of your globe, beach ball, or Lénárt sphere. Using string, find the shortest distance between your two points. The shortest distance is an arc. Trace this arc. This arc lies on a great circle which is the set of points on the surface formed by the intersection of the sphere and a plane passing through the center of the sphere. Extend your arc to find the great circle that contains it.
2. Is there more than one great circle passing through two points on your sphere? Explain.
3. Find several examples of great circles on your sphere. Are great circles infinite in length? Why or why not?
4. Can great circles be parallel? Explain.
5. Locate three non-collinear points on your sphere. To form an elliptic triangle through your three points, draw the three great circles that connect pairs of these points. Measure each angle of your elliptic triangle. What is the sum of the measures of the three angles? Can you find a triangle whose three angles add up to $270^{\circ}$ ? Can you find a triangle whose three angles add up to $360^{\circ}$ ? Can you find a triangle whose three angles add up to more than $360^{\circ}$ ?
6. Can you find similar triangles on your sphere that are not congruent? Explain.
7. Restate the theorem "The sum of the measures of the angles of a triangle is $180^{\circ "}$ so that it applies to elliptic space.
8. Is there a relationship between the size of a triangle on your sphere and the sum of the measures of its angles? Explain.
9. Restate Euclid's first five postulates so that they apply to the sphere.
10. Compare the Gaussian curvature of an elliptic surface to the sum of the angles of a triangle on its surface.
11. Describe the Gaussian curvature of an elliptic surface. Does Euclid's parallel postulate hold for that surface?

## Euclid's First Five Postulates in Hyperbolic Space

Overview: In this activity, participants explore Euclid's first five postulates in hyperbolic space.

## Objective: TExES Mathematics Competencies

III.012.A. The beginning teacher understands axiomatic systems and their components (e.g., undefined terms, defined terms, theorems, examples, counterexamples).
III.012.G. The beginning teacher compares and contrasts the axioms of Euclidean geometry with those of non-Euclidean geometry (i.e., hyperbolic and elliptic geometry).
V.018.C. The beginning teacher uses inductive reasoning to make conjectures and uses deductive methods to evaluation the validity of conjectures.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.
V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

## Geometry TEKS

b.1.A. The student develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems.
b.1.C. The student compares and contrasts the structures and implications of Euclidean and non-Euclidean geometries. b.3.C. The student demonstrates what it means to prove mathematically that statements are true.
b.3.D. The students uses inductive reasoning to formulate a conjecture.

Background: Participants should have prior knowledge of Euclid's first five postulates in Euclidean space.

Materials: compass, straightedge, colored pencils, computer lab and/or computer with projector, NonEuclid (available at http://cs.unm.edu/~joel/NonEuclid/NonEuclid.html)

New Terms: hyperbolic geometry or hyperbolic space, hyperbolic lines, Poincaré disk

## Procedures:

Have participants recall the negation of Euclid's fifth postulate "Through a point not on a line, there exists more than one line parallel to the line."

Hyperbolic geometry, which is also referred to as Lobachevskian geometry, was independently discovered by Nicholas Lobachevsky, Janos Bolyai, and Carl Fredrich Gauss. Lobachevsky was the first to publish his work on hyperbolic geometry (1829).

Hyperbolic geometry is a geometry on a curved surface that resembles a saddle, a bicycle seat, a witch's hat, ruffled lettuce, a hyperboloid of one sheet, a hyperbolic paraboloid, or a pseudosphere. The illustrations below are examples of hyperbolic surfaces.


## Is there a mathematical model for hyperbolic space?

There are several models for hyperbolic space. We will consider the model that Henri Poincaré (1854-1912) created, called the Poincaré disk. Poincaré represented points in hyperbolic space as points in a circular disk. The circular boundary of this disk is not included; the points on this circle are considered to be infinitely far away from points in the disk.

As in elliptic geometry, the discussion of geodesics for hyperbolic space, in general, is beyond the scope of this unit. As a result, we will restrict the discussion to Poincare's disk, a hyperbolic surface of constant negative Gaussian curvature. We will refer to a line in Poincaré's model as a hyperbolic line. There are two types of hyperbolic lines. The first type is any diameter of the Poincaré disk. These hyperbolic lines pass through the center of the Poincare disk. The second type is determined by Euclidean circles that meet the circular boundary of the Poincaré disk at right angles. The portion of such a circle inside the Poincaré disk is a hyperbolic line. Notice that a hyperbolic line of the second type cannot pass through the center of the Poincaré disk. Notice also that the endpoints of the segment or arc that is a hyperbolic line lie on the circular boundary, and are therefore not part of the hyperbolic line. Thus, we may consider hyperbolic lines as open at the two endpoints.

The shortest distance between two points on Poincarés disk lies on a hyperbolic line. Therefore the hyperbolic lines are the geodesics on Poincare's disk. This model of hyperbolic space distorts distances. The illustration below shows four hyperbolic lines in Poincare's model of hyperbolic space intersecting at point $A$. One of the hyperbolic lines
passes through the center of the circle and is straight. The other three do not pass through the center of the circle and are parts of Euclidean circles. All four, by definition, are orthogonal to the boundary of the disk.


The endpoints of hyperbolic lines are not part of the hyperbolic lines but represent the hyperbolic lines at infinity (remember that points on the circle are considered infinitely far away). To measure angles formed by a pair of these hyperbolic lines, simply measure the angles formed by the tangents to the hyperbolic lines at their points of intersection. Distance in this model is more complicated. As lines in Euclidean space, hyperbolic lines in Poincare's model of hyperbolic space have infinite length, because you compress distance as you get closer to the boundary of the disk. Thus segments that look congruent may have different lengths, depending to their proximity to the center of the circle.

To visualize how distance works in this model, look at Escher's Circle Limit patterns. Imagine that all the figures are congruent, but they appear to get smaller at the edge of the disk, because they are far away.


Circle Limit IV, Escher from Symmetry, Shape, and Space © 2002 (p. 334) with permission from Key Curriculum Press.

The following demonstration is taken from Discovering Geometry: An Inductive Approach: $3^{\text {rd }}$ Edition, Teacher's Edition © 2003, pp. 719 with permission from Key Curriculum Press.

You may want to demonstrate the following model of Poincare's disk. Take a circle made of rubber with the edges stretched out or ruffled like the edges of a lasagna noodle. Pull the opposite edges tight; draw a line with a straight-edge; then release the edges. The line will appear curved (a hyperbolic line).

Have participants complete the activity sheet using NonEuclid, an interactive web site at http://cs.unm.edu/~joel/NonEuclid/NonEuclid.html that allows participants to explore hyperbolic geometry.

The following are adapted from Discovering Geometry: An Inductive Approach: $3^{\text {rd }}$ Edition, © 2003, pp. 718-720 with permission from Key Curriculum Press.

1. Is there more than one hyperbolic line passing through two points on Poincaré's disk? Explain.
No, there can only be one hyperbolic line that passes through two points.
Participants should note that an infinite number of hyperbolic lines can be drawn through a single point.
2. Are hyperbolic lines infinite in length? Why or why not?

Hyperbolic lines are infinite in length since their endpoints lie on the circle and points on the circle are considered infinitely far away.
Hyperbolic lines may look finite in length but their endpoints lie on the boundary of Poincare's disk which is at infinity.
3. Can hyperbolic lines be parallel? Illustrate with a picture and explain. Hyperbolic lines can be parallel, but they are never equidistant. In the illustration below, the intersecting hyperbolic lines are parallel to the top two hyperbolic lines, and the top two hyperbolic lines are also parallel to each other.


Have participants observe that if two hyperbolic lines intersect, they intersect in exactly one point. Therefore, hyperbolic lines intersect in exactly one point or in no points.
4. Draw two hyperbolic lines that are perpendicular to the same hyperbolic line. Are they parallel?


In the illustration above, the two hyperbolic lines passing through points $R$ and $P$ are perpendicular to the hyperbolic line $m$. These two hyperbolic lines are parallel to each other. Notice that they are not, however, equidistant. Also notice that hyperbolic lines curve away from the center of the circle unless they pass through its center (straight lines).
5. Draw four hyperbolic lines passing through the same point that are parallel to (not intersecting) a fifth hyperbolic line.
In the illustration below, the four hyperbolic lines that pass through point $P$ are parallel to hyperbolic line $m$.

6. Locate three non-collinear points on Poincaré's disk. To form a hyperbolic triangle, draw the three hyperbolic lines that connect pairs of points. Measure each angle of your triangle. What is the sum of the measures of the three angles of your triangle? Can you find a triangle whose three angles add up to $180^{\circ}$ ? Can you find a triangle whose three angles add up to more than $180^{\circ}$ ? Can you find a triangle whose three angles add up to $0^{\circ}$ ?
Answers will vary as the sum of the measures of the angles of a hyperbolic triangle depends upon the size of the triangle. Very small triangles (which appear Euclidean) have the sum of their angles very close to $180^{\circ}$. The larger the triangle, the smaller the sum of its angles; however, the sum is always less than $180^{\circ}$. The sum of the measures of the angles of a triangle whose vertices lie on the boundary of the circle is $0^{o}$.

7. Can you find similar triangles on Poincaré's disk that are not congruent? Explain. In hyperbolic space, similar triangles must be congruent.
8. Restate the theorem "The sum of the measures of the angles of a triangle is $180^{\circ}$ " so that it applies to hyperbolic space.

The sum of the measures of the angles of a hyperbolic triangle is always less than $180^{\circ}$.
9. Restate Euclid's first five postulates so that they apply to Poincare's disk.

- For any two distinct points, there is one hyperbolic line that contains them.
- Any segment may be extended indefinitely along a hyperbolic line.
- Given a point (center) and a distance (radius), a circle can be drawn. [Unchanged from Euclidean space]

Have participants investigate what happens when the center of the circle is near the edge of Poincare's disk. Ask them to measure several radii of such a circle to observe that they are indeed congruent even though they do not appear to be. Remind participants that distance on Poincare's disk is distorted and that congruent lengths may not appear to be congruent.

- All right angles are congruent. [Unchanged from Euclidean space]
- Through a point not on a hyperbolic line, there are an infinite number of hyperbolic lines parallel to it. In the illustration below, the four hyperbolic lines passing through point $P$ are parallel to the hyperbolic line $m$.


10. Compare the Gaussian curvature of a hyperbolic surface to the sum of the angles of a triangle on its surface.
A surface with negative Gaussian curvature is hyperbolic. The sum of the measures of the angles of a triangle on its surface is less than $180^{\circ}$.
11. Describe the Gaussian curvature of a hyperbolic surface. Does Euclid's parallel postulate hold for its surface?
A surface with a negative Gaussian curvature is hyperbolic. The sum of the measures of the angles of a triangle on its surface is less than $180^{\circ}$. The parallel postulate does not hold on that surface. For example, on Poincaré's disk, through a point not on a hyperbolic line, there are an infinite number of hyperbolic lines parallel to it.

Remind participants to add the terms hyperbolic geometry, Poincaré disk and hyperbolic lines to their glossaries.

## Euclid's First Five Postulates in Hyperbolic Space

The illustration below shows several hyperbolic lines on Poincare's disk, a model of hyperbolic space. The shortest distance between two points on Poincare's disk lies on a hyperbolic line. A hyperbolic line is the set of points on the hyperbolic surface that meets the perimeter of the surface at right angles at each endpoint.


To visualize how distance works in this model, look at Escher's Circle Limit patterns. Imagine that all the figures are congruent, but they appear to get smaller at the edge of the disk because they are far away. Points on the circle are infinitely far away.


Circle Limit IV, Escher from Symmetry, Shape, and Space © 2002 (p. 334) with permission from Key Curriculum Press.

Complete 1-11 using NonEuclid, an interactive web site at http://cs.unm.edu/~joel/NonEuclid/NonEuclid.html Draw sketches to support your answers when appropriate.

1. Is there more than one hyperbolic line passing through two points on Poincaré's disk? Explain.
2. Are hyperbolic lines infinite in length? Why or why not?
3. Can hyperbolic lines be parallel? Illustrate with a picture and explain.
4. Draw two hyperbolic lines that are perpendicular to the same hyperbolic line. Are they parallel?
5. Draw four hyperbolic lines passing through the same point that are parallel to (not intersecting) a fifth hyperbolic line.
6. Locate three non-collinear points on Poincaré's disk. To form a hyperbolic triangle, draw the three hyperbolic lines that connect pairs of points. Measure each angle of your triangle. What is the sum of the measures of the three angles of your triangle? Can you find a triangle whose three angles add up to $180^{\circ}$ ? Can you find a triangle whose three angles add up to more than $180^{\circ}$ ? Can you find a triangle whose three angles add up to $0^{\circ}$ ?
7. Can you find similar triangles on Poincare's disk that are not congruent? Explain.
8. Restate the theorem "The sum of the measures of the angles of a triangle is $180^{\circ "}$ so that it applies to hyperbolic space.
9. Restate Euclid's first five postulates so that they apply to Poincare's disk.
10. Compare the Gaussian curvature of a hyperbolic surface to the sum of the angles of a triangle on its surface.
11. Describe the Gaussian curvature of a hyperbolic surface. Does Euclid's parallel postulate hold for its surface?

## Visualizing Three Different Geometries

Overview: In this activity, participants construct models of elliptic and hyperbolic surfaces in order to compare them to the Euclidean plane.

## Objective: TExES Mathematics Competencies

III.012.A. The beginning teacher understands axiomatic systems and their components (e.g., undefined terms, defined terms, theorems, examples, counterexamples).
III.012.G. The beginning teacher compares and contrasts the axioms of Euclidean geometry with those of non-Euclidean geometry (i.e., hyperbolic and elliptic geometry).
V.018.C. The beginning teacher uses inductive reasoning to make conjectures and uses deductive methods to evaluation the validity of conjectures.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.
V.019.F. The beginning teacher uses appropriate mathematical terminology to express mathematical ideas.

## Geometry TEKS

b.1.A. The student develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems.
b.1.B. Through the historical development of geometric systems, the student recognizes that mathematics is developed for a variety of purposes.
b.1.C. The student compares and contrasts the structures and implications of Euclidean and non-Euclidean geometries.

Background: Participants should have prior experience tiling the plane with equilateral triangles.

Materials: equilateral triangle paper with side length of at least one inch (several sheets per participant-provided in the Appendix), scissors, tape

## New Terms:

## Procedures:

Have participants work in pairs to construct the models of elliptic and hyperbolic space. By constructing these models, participants will gain new insights about Euclidean, elliptic, and hyperbolic surfaces. They will observe that their models of elliptic and hyperbolic space cannot tile the plane.

Each pair of participants should have several sheets of equilateral triangle paper, scissors, and tape. For the two constructions, you may copy the equilateral triangle paper which can be found in the Appendix.

Ask the group the question "How many equilateral triangles meet at a common vertex in the equilateral triangle paper?"
Participants should respond that six equilateral triangles meet at a common vertex. For Euclidean space, six equilateral triangles meet together at a common point without gaps or without overlapping. Participants should note that each angle of an equilateral triangle has a measure of $60^{\circ}$, and six angles meet to form a complete revolution of $360^{\circ}$ on a flat surface. This surface has zero Gaussian curvature.

Ask the group the questions "What do you think will happen if you reduce or increase the number of equilateral triangles that meet at a common vertex?" and "What type of surface will you have?"
Let participants offer their ideas. The two constructions on the activity sheet will answer these two questions.

Have pairs of participants construct the two models described on the activity sheet. Carefully monitor participants, making sure that the first model they build has exactly five equilateral triangles meeting at each vertex and that their second model has exactly seven equilateral triangles meeting at each vertex. Participants may wish to cut out hexagons, half-hexagons, or individual equilateral triangles from the equilateral triangle paper to make their models. One participant should hold the triangles together as the other participant tapes the triangles together to make the models.

1. Fit and tape five equilateral triangles together so that they meet at a common vertex. To do this, use the equilateral triangle paper provided. One way to do this is to cut out a hexagon from the triangle paper. Remove one of the equilateral triangles. Tape the remaining five equilateral triangles back together so that you have all five triangles meeting at the common vertex with no gap where the sixth triangle was removed. This will result in a three-dimensional figure. Continue the process of fitting and taping exactly five equilateral triangles out from the original vertex until you cannot add more triangles to your model. Make sure only five equilateral triangles meet at each vertex. Alternately, you may also choose to cut out and tape individual equilateral triangles so that five triangles fit together at a common vertex. Describe the surface that you get.
You do not get a flat surface. You get an icosahedron since only twenty equilateral triangles can fit together using this construction process. This construction produces a surface with positive Gaussian curvature.

For this model of elliptic space, participants taped together five equilateral triangles coming together at each vertex rather than the six that came together at a common vertex in the equilateral triangle paper. Equilateral triangle paper can be used to represent Euclidean space since it is flat. The surface that participants constructed here was not flat. It also could not be extended indefinitely from the original vertex. Five
equilateral triangles meeting at a common vertex leave a gap; therefore this construction cannot tile the plane. There is a deficit in angle measure when the five $60^{\circ}$ angles come together at a common vertex. A flat surface is not created. The name for elliptic geometry as well as the name for the ellipse comes from the Greek word that means deficient.

For an alternate model of elliptic geometry, you may choose to carefully peel the rind from an orange or tangerine to observe that the rind, when flattened, leaves gaps. Since there are gaps between the pieces of the rind, the rind cannot tile the plane.
2. Fit and tape seven equilateral triangles so that they meet at a common vertex. To do this, use the equilateral triangle paper provided. One way to do this is to cut out a hexagon from the triangle paper. Cut a side of one of the equilateral triangles from the center out, then insert and tape another equilateral triangle so that seven equilateral triangles are taped together at the vertex. Continue the process of fitting and taping exactly seven equilateral triangles together out from the original vertex for at least six more vertices. Make sure that seven equilateral triangles meet at each vertex and that at each edge only two triangles meet. Alternately you may cut out and tape individual equilateral triangles together so that seven triangles fit together at a common vertex. Describe the surface that you get.
You do not get a flat surface, but it is different from the first construction, as this one is "floppy" instead of "bulging out." This construction produces a surface with negative Gaussian curvature.

For this model of hyperbolic space, participants taped together seven equilateral triangles at each vertex. The surface that is constructed is not flat, but it does not resemble the first construction, as this one is floppy and can be extended out infinitely from the original vertex. Seven equilateral triangles meeting at a common vertex are too many to exactly tile the plane. There is an excess in angle measure coming together at a common vertex to create a flat surface. The name for hyperbolic geometry as well as the name for the hyperbola comes from the Greek word that means excessive.

Participants may work together with other groups to make as large a model of hyperbolic space as time permits.
3. Explain why the two models that you constructed are different and how they compare to the Euclidean plane.
It takes six equilateral triangles meeting at a vertex to form a flat surface. The sum of the measures of the six angles is $360^{\circ}$, which is a complete revolution in Euclidean space.

Having too few triangles meeting at a common point (as in the first construction) leaves a gap causing the surface to bulge out. The sum of the measures of the five angles at each vertex is $300^{\circ}$, which is not enough to form a complete revolution in Euclidean space. This surface has positive Gaussian curvature.

Having too many triangles meeting at a point (as in the second construction) causes an excess of surface area, causing the surface to ruffle and be floppy. The sum of the measures of the seven angles at each vertex is $420^{\circ}$, which is too much to form a complete revolution in Euclidean space. This surface with has negative Gaussian curvature.

Neither of the two models constructed tiles the plane. The elliptic surface comes apart and leaves gaps when flattened, whereas the hyperbolic surface when flattened overlaps onto itself.

Moreover, the elliptic model has a finite surface area, whereas the hyperbolic model continues infinitely.

The two constructions give participants a concrete look at elliptic and hyperbolic surfaces.

If time permits, engage participants in a discussion based on the following questions:

- What is the geometry of our universe?
- Why did early explorers think the earth was flat?
- Why might space travelers to the moon think the earth is elliptic?
- Could our universe be hyperbolic?

It's all about a person's vantage point in the universe!
For additional information for constructing models of hyperbolic space, you may want to refer to Professor Diane Hoffoss' web site (formerly of the Rice University Mathematics Department) at http://home.sandiego.edu/~dhoffoss/rusmp/

## Visualizing Three Different Geometries

1. Fit and tape five equilateral triangles together so that they meet at a common vertex. To do this, use the equilateral triangle paper provided. One way to do this is to cut out a hexagon from the triangle paper. Remove one of the equilateral triangles. Tape the remaining five equilateral triangles back together so that you have all five triangles meeting at the common vertex with no gap where the sixth triangle was removed. This will result in a three-dimensional figure. Continue the process of fitting and taping exactly five equilateral triangles out from the original vertex until you cannot add more triangles to your model. Make sure only five equilateral triangles meet at each vertex. Alternately, you may also choose to cut out and tape individual equilateral triangles so that five triangles fit together at a common vertex. Describe the surface that you get.
2. Fit and tape seven equilateral triangles so that they meet at a common vertex. To do this, use the equilateral triangle paper provided. One way to do this is to cut out a hexagon from the triangle paper. Cut a side of one of the equilateral triangles from the center out, then insert and tape another equilateral triangle so that seven equilateral triangles are taped together at the vertex. Continue the process of fitting and taping exactly seven equilateral triangles together out from the original vertex for at least six more vertices. Make sure that seven equilateral triangles meet at each vertex and that at each edge only two triangles meet. Alternately you may cut out and tape individual equilateral triangles together so that seven triangles fit together at a common vertex. Describe the surface that you get.
3. Explain why the two models that you constructed are different and how they compare to the Euclidean plane.

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## The Geometer's Sketchpad Unit 1 Meet Geometer's Sketchpad

Overview: In this unit, participants become familiar with the menus and tools of The Geometer's Sketchpad (Sketchpad) dynamic geometry program through exploration and construction.

## Objective: TExES Mathematics Competencies

III.012.E. The beginning teacher describes and justifies geometric constructions made using compass and straightedge, reflection devices, and other appropriate technologies.
VI.020.D. The beginning teacher understands how learning may be enhanced through the use of manipulatives, technology, and other tools.

## Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.

Materials: The Geometer's Sketchpad program, sample sketch: Unit 1-Sample Sketch

## Procedures:

It is important that the presenter guide the participants through the first few Sketchpad training units. Remember to keep the pace of the activity slow, with the understanding that many of the participants will be unfamiliar with the technology. Adjust the pace of the units as needed. Extension activities are included in some units to allow for differentiation. Every participant may not complete every unit. For an intensive tutorial on the use of Sketchpad see Region IV ESC Geometer's Sketchpad Tutorial at http://www.esc4.net/math/sketchpadtutorial. Macintosh users need to use the menu commands in places where PC users use the right click shortcut. The first unit will provide the presenter with an informal assessment of participants' ability with the technology.

## Part A

The presenter guides participants through a tour of the menus and tools of Sketchpad. Participants should have access to the Sketchpad program to participate in the tour with the presenter.

## Part B

The presenter guides the participants through the setup of a multi-page document. Participants perform two very important constructions with the software, a line parallel to another line through a point and a line perpendicular to a line through a point. Use the Unit 1-Sample Sketch as a model for participants.

## Geometry Unit

## Part C

The presenter guides participants through the creation of a happy face. The purpose of this activity is to apply the tools of Sketchpad in a fun manner. Participants draw elementary geometric figures which require a basic knowledge of the tools of Sketchpad. Use the Unit 1Sample Sketch as a model for participants.

Part A

(4) The Geometer's Sketchpad - [Untitled 1]
(4) File Edit Display Construct Transform Measure Graph Window Help


(4.) The Geometer's Sketchpad - [Until


4. The Geometer's Sketchpad - [Untitied 1]
(a) File Edit Display Construct Transform Measure Graph Window Help


The Display menu contains commands for controlling the look of a sketch. A user can control the width of a line, the color of an object, hide or show objects, trace objects, and hide and show the text editing toolbar.
(4) The Geometer's Sketchpad - [Untitled 1]




## (4) The Geometer's Sketchpad - [Untitied 1]

Use the Selection Arrow tool to select objects, rotate objects, and translate objects. To change the Selection Arrow tool, click and hold on the arrow, then select the desired tool. The most commonly used tool is the arrow, or Selection Arrow. Remember to click on the Selection Arrow tool whenever you finish drawing an object. Pressing the ESC key is a shortcut to picking up the Selection Arrow tool.


Draw a few points and then press the ESC key to select the arrow tool and deselect all objects. Click on a few points to select them and then select hide points from the display menu. Select show all hidden from the display menu. Finally select the points by clicking and dragging the selection arrow to create a box around the points then press delete to delete them (see figure below). Use the Selection Arrow to select points.

(©) The Geometer's sketctipad - [Untited 1]
(4) File Edit Display Construct Transform Measure Graph Window Help


Use the Straightedge tool to draw segments, rays, and lines.
Click and hold on the Straightedge tool to select the Segment, Ray, or Line tool.

Draw a few segments, rays and lines and then press the ESC key to select the arrow tool. Click and drag various points to change the length and direction of the segments and the direction of the rays and lines.

## 4. The Geometer's Sketchpad - [Untitied 1] <br> (4) File Edit Display Construct Transform Measure Graph Window Help



Label the endpoints of a segment you have drawn. Label the segment itself. Double click a label and change the label (see figure below).



## Part B



- Open The Geometer's Sketchpad.
- From the File menu, choose New Sketch.
- From the File menu, choose Document Options.
- Name the first page Parallel, then click Add Page, choose Blank Page and name that page Perpendicular.
- Click OK.
- From the Edit menu choose Preferences.
- Make sure that the settings are what you want, then click OK.

- Click on the Parallel tab in the lower left-hand section of the window to select the Parallel page.
- Use the Straightedge tool to draw a line (hold down Shift while drawing to draw a horizontal line).
- Use the Point tool to draw a point not on the line.
- Press ESC.
- Select the line and the point not on the line.
- From the Construct menu, choose Parallel Line.

- Click on the Perpendicular tab at the lower left hand section of the window to select the Perpendicular page.


## © The Geometer's Sketchpad - [Untitited 1 - Parallel]

(4) File Edit Display Construct Transform Measure Graph Window Help


- Use the Straightedge tool to draw a line (hold down Shift while drawing to draw a horizontal line).
- Use the Point tool to draw a point not on the line.
- Press ESC.

- Select the line and the point not on the line.
- From the Construct menu, choose Perpendicular Line.


Part C



- Draw a circle.
- Draw a vertical line through the center.
- Draw a triangle for the right eye using the Segment tool to construct three segments.
- Press ESC to deselect all objects.
- Select the three vertices of the triangles.
- From the Construct menu, choose Triangle Interior.
- Press ESC.
- Double click the vertical line.
- Select all vertices, edges, and the interior of the triangle.
- From the Transform menu, choose Reflect.
- Press ESC.
- Draw a small circle centered at the original circle center.
- Select the circle.
- From the Construct menu, choose Circle Interior.
- Press ESC.
- Right click on the circle interior and choose Color (or choose Color from the Display menu) to change the color.
- Draw two overlapping circles as shown in the figure at left.
- Select both circles.
- From the Construct menu, choose Intersections.



## The Geometer's Sketchpad Unit 2 <br> Transformations

Overview: In this unit, participants learn various techniques for performing transformations and creating interactive sketches to explore the properties of transformations.

## Objective: TExES Mathematics Competencies

III.012.E The beginning teacher describes and justifies geometric constructions made using compass and straightedge, reflection devices, and other appropriate technologies. III.012.F. The beginning teacher demonstrates an understanding of the use of appropriate software to explore attributes of geometric figures and to make and evaluate conjectures about geometric relationships.
III.014.A. The beginning teacher identifies transformations (i.e., reflections, translations, glide-reflections, rotations, dilations) and explores their properties.
V.019.E. The beginning teacher understands the use of visual media, such as graphs, tables, diagrams, and animations, to communicate mathematical information.
VI.020.D. The beginning teacher understands how learning may be enhanced through the use of manipulatives, technology, and other tools.

## Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.D. The student uses inductive reasoning to formulate a conjecture.
e.2.A. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties of parallel and perpendicular lines.
e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.
f.1. The student uses similarity properties and transformations to explore and justify conjectures about geometric figures.

Materials: $\quad$ The Geometer's Sketchpad program, sample sketches: Unit 2-Sample Sketch 1, Unit 2Sample Sketch 2, and Unit 2-Sample Sketch 3

## Procedures:

When directions require participants to select an object in The Geometer's Sketchpad (for example, to select $\triangle A B C$ ), the participant should select all vertices, sides, and, if appropriate, the interior of the triangle.

Journal prompts are included for participants to record observations and explanations for many of the investigations.

## Part A

The presenter guides participants through various methods of performing transformations in The Geometer's Sketchpad. These methods increase the interactivity of the sketch and allow students to use inductive reasoning to formulate conjectures. Participants work through the Unit 2-Sample Sketch 1 beginning with the introduction page and move through the pages in order.

## Part B

Participants use their knowledge of translations and The Geometer's Sketchpad to create an interactive sketch that will allow their students to formulate conjectures about the properties of translations. Use the Unit 2-Sample Sketch 2 to model a sample final product for participants.

## Part C

Participants investigate the Unit 2-Sample Sketch 3 and respond to the journal prompts.

## Part A

Work through the pages in the Unit 2-Sample Sketch 1 beginning with the introduction page and move through the pages in order.

## Part B

Create a sketch that students can use to investigate the properties of a translation.


- Start The Geometer's Sketchpad.
- Open a new sketch.
- From the Graph menu, choose Show Grid.
- From the Graph menu, choose Snap Points.

- Draw a polygon in the first quadrant.
- Label the vertices of the polygon.
- Select the vertices of the polygon.
- From the Construct menu, choose Polygon Interior.
- Measure the coordinates of the vertices of the polygon.
- Measure the side lengths of the polygon by selecting the endpoints then from the Measure menu choose Distance.

- Draw a translation vector $\overline{G H}$.
- Select $G$ then $H$.
- From the Transform menu, choose Mark Vector.
- Select the polygon.
- From the Transform menu, choose Translate. Click on the Marked Vector button.
- Click Translate.
- Measure the coordinates, side lengths and area of the image. (To measure the area, click on the polygon interior then choose Area from the Measure menu.)
- Click ESC.
- Measure the area of the pre-image.
- Use the Segment tool to draw the segments connecting corresponding vertices of the preimage and image.


## Journal Entry

Record your observations and explanations for the following:

- Drag the vertices of the pre-image or image.
- Drag $H$, the endpoint of the translation vector.


## Part C

Investigation of the angles formed when parallel lines are cut by a transversal.


## Journal Entry

Record your observations and explanations for the following:

- Click the Translate button.
- $\operatorname{Drag} A, B, C, D$.


## The Geometer's Sketchpad Unit 3 <br> Triangles and Quadrilaterals

Overview: In this unit, participants construct points of concurrency for triangles and create interactive sketches to explore the properties of points of concurrency and quadrilaterals.

## Objective: TExES Mathematics Competencies

III.011.A. The beginning teacher applies dimensional analysis to derive units and formulas in a variety of situations (e.g., rates of change of one variable with respect to another) and to find and evaluate solutions to problems.
III.012.E. The beginning teacher describes and justifies geometric constructions made using compass and straightedge, reflection devices, and other appropriate technologies. III.012.F. The beginning teacher demonstrates an understanding of the use of appropriate software to explore attributes of geometric figures and to make and evaluate conjectures about geometric relationships.
III.013.A. The beginning teacher analyzes the properties of polygons and their components.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
III.014.C. The beginning teacher uses transformations to explore and describe reflectional, rotational, and translational symmetry.
III.014.E. The beginning teacher applies concepts and properties of slope, midpoint, parallelism, perpendicularity, and distance to explore properties of geometric figures and solve problems in the coordinate plane.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.
V.019.E. The beginning teacher understands the use of visual media, such as graphs, tables, diagrams, and animations, to communicate mathematical information. VI.020.D. The beginning teacher understands how learning may be enhanced through the use of manipulatives, technology, and other tools.

## Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties to polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
d.2.C. The student develops and uses formulas including distance and midpoint.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.
e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.
f.1. The student uses similarity properties and transformations to explore and justify conjectures about geometric figures.

Materials: The Geometer's Sketchpad program, sample sketches: Unit 3-Sample Sketch 1, Unit 3Sample Sketch 2

## Procedures:

Journal entry prompts are included at the end of Part A and Part B. Participants may wish to complete journal entries as they progress through each investigation.

## Part A

Participants use their knowledge of The Geometer's Sketchpad to create a multi-page sketch that allows their students to formulate conjectures about points of concurrency in triangles. Participants should be able to apply their knowledge gained in previous The Geometer's Sketchpad training activities and require less guidance in the creation of this sketch. Use the Unit 3-Sample Sketch 1 to model a sample final product for participants.

## Part B

Participants use their knowledge of The Geometer's Sketchpad to create a multi-page sketch that allows their students to formulate conjectures about properties of quadrilaterals. Participants should be able to apply their knowledge gained in previous The Geometer's Sketchpad training activities and require less guidance in the creation of this sketch. Use the Unit 3-Sample Sketch 2 to model a sample final product for participants.

## Part C

Use the quadrilateral that you selected in the Unit 4 - Informal Logic/Deductive Reasoning, Alternate Definitions activity. Use two of the properties to construct the quadrilateral with The Geometer's Sketchpad. Drag any one of the figure's vertices and it should remain that figure, i.e. a rectangle remains a rectangle. The figure chosen for the sample construction is a rectangle.
The two properties chosen are that opposite sides of a rectangle are parallel and consecutive sides of a rectangle are perpendicular.


- Draw a segment $\overline{A B}$.
- Construct a line parallel to $\overline{A B}$ through C not on the segment.
- Construct lines perpendicular to $\overline{A B}$ through the endpoints of $\overline{A B}$.
- Construct intersections $D$ and $E$.
- Draw $\overline{A D}, \overline{D E}$, and $\overline{E B}$. Hide $\overparen{A D}, \overleftrightarrow{B E}, \overleftrightarrow{D E}$ and $C$.
- Drag vertices $A, B, D$, or $E$ to show that $A B D E$ remains a rectangle.



## Part A

Create a multi-page sketch that allows students to formulate conjectures about points of concurrency in triangles. Apply your knowledge gained in previous Sketchpad training activities to create this sketch. Use the Unit 3-Sample Sketch 1 as a guide.


- Start Sketchpad.
- Open a new sketch.
- Use the Document Options command to create a six page document with page titles Introduction, Incenter, Circumcenter, Orthocenter, Centroid, and Euler Line.
- Click on the Introduction tab.
- Create a set of instructions for the activity. (These can be modified later.)

- Click on the Incenter tab.
- Draw a triangle.
- Construct the angle bisectors of each angle of the triangle.
- Use the Arrow tool to click on the intersection of the angle bisectors to construct a point of intersection called the incenter.
- Label the vertices of the triangle and the incenter.
- Construct a perpendicular line to a side through the incenter.
- Construct the point of intersection of the side and the perpendicular line.
- Draw a segment from the incenter to the point of intersection just created.
- Hide the perpendicular line.
- Label the segment just drawn radius.
- Construct the inscribed circle, using the incenter as the center and the segment labeled radius as the radius.
- Measure the distances from the incenter to the sides of the triangle.
- Write instructions.
- Create Hide/Show action buttons for the angle bisectors, the inscribed circle, and the measurements.
- Create a link action button that will link back to the introduction page.

- Click on the Circumcenter tab.
- Draw a triangle.
- Construct the midpoints of the sides of the triangle.
- Construct the perpendicular bisector to each side of the triangle.
- Use the Arrow tool to click on the intersection of the perpendicular bisectors to construct a point of intersection called the circumcenter.
- Label the vertices of the triangle and the circumcenter.
- Draw a segment from the circumcenter to a vertex.
- Label the segment just drawn radius.
- Construct the circumscribed circle, using the circumcenter as the center and the segment labeled radius as the radius.
- Measure the distances from the circumcenter to the vertices of the triangle.
- Write instructions.
- Create Hide/Show action buttons for the perpendicular bisectors, the circumscribed circle, and the measurements.
- Click on the Orthocenter tab.
- Draw a triangle.
- Construct the altitude to each side of the triangle.
- Use the Arrow tool to click on the intersection of the altitudes to construct a point of intersection called the orthocenter.
- Label the vertices of the triangle and the orthocenter.
- Draw the segments from the orthocenter to each vertex of the triangle creating three small triangles.
- Construct the altitude to each side of each of the small triangles formed.
- Write instructions.
- Create Hide/Show action buttons for both sets of altitudes and the segments connecting the orthocenter to the vertices of the triangle.
- Create a link action button that will link back to the introduction page.

- Click on the Euler Line tab.
- Draw and label a triangle.
- Construct the incenter, the circumcenter, the orthocenter, and the centroid of the triangle and label them.
- Hide all construction lines and points except for the triangle and points of concurrency.
- Construct the Euler Line by constructing a line through the orthocenter and the circumcenter and label it.
- Calculate the ratio of the distance from the orthocenter to the centroid to the distance from the centroid to the circumcenter.
- Measure the lengths of the sides of the triangle and measure its angles.
- Write instructions.
- Create Hide/Show action buttons for the points of concurrency, the Euler Line, and the measurements.


## Journal Entry

Record your observations and explanations for

- Create a link action button that will link back to the introduction page. the following:
Incenter
- Drag the vertices of the triangle and investigate the location and properties of the incenter.

Circumcenter

- Drag the vertices of the triangle and investigate the location and properties of the circumcenter.

Orthocenter

- Drag the vertices of the triangle and investigate the location and properties of the orthocenter. Centroid
- Drag the vertices of the triangle and investigate the location and properties of the centroid.
- Explain why the areas of the triangles formed by the medians are equal.


## Euler Line

- Is there a triangle where all of the points of concurrency are on the Euler Line?
- Is there a triangle where the points of concurrency are in the same location?


## Part B

Create a multi-page sketch that will allow students to formulate conjectures about points of concurrency in triangles. Apply your knowledge gained in previous The Geometer's Sketchpad training activities to create this sketch. Use the Unit 3-Sample Sketch 2 as a guide.


- Start The Geometer's Sketchpad.
- Open a new sketch.
- Use the Document Options command to create a five page document with page titles; Instructions, Right Triangle Reflections, Acute/Obtuse Triangle Reflections, Rotate a Triangle, and Truncate a Triangle Vertex.
- Click on the Instructions tab.
- Create a set of instructions for the activity. (These can be modified later.)


- Click on the Right Triangle Reflections tab.
- Construct right $\triangle A B C$ with right angle B.
- Reflect $A, \overline{A C}$ and $\overline{A B}$ over the line that contains $\overline{B C}$.
- Draw $\overline{A^{\prime} B}$ and $\overline{A^{\prime} C}$.
- Reflect $C, \overline{B C}, \overline{A C}$, and $\overline{A C}$ over the line that contains $A$ and $A^{\prime}$.
- Draw $\overline{A C^{\prime}}, \overline{B C^{\prime}}$, and $\overline{A^{\prime} C^{\prime}}$.
- Create Hide/Show buttons to hide $\Delta A^{\prime} B C, \triangle A C^{\prime} A^{\prime}$.
- Name the buttons "Reflect triangle $A B C$ over the line containing $B$ and $C$ " and "Reflect triangle $A C A$ ' over the line containing $A$ and $A^{\prime}{ }^{\prime \prime}$.
- Measure all segment lengths and all angles of the quadrilateral.
- Create Hide/Show action buttons for the constructed triangles and the measurements.
- Click on the Acute/Obtuse Triangle Reflections tab.
- Construct $\triangle A B C$.
- Reflect $A$ over the line that contains $\overline{B C}$
- Draw $\overline{A^{\prime} C}$ and $\overline{A^{\prime} B}$.
- Draw $\overline{A A^{\prime}}$ and label $D$ the intersection of the diagonals.
- Create a Hide/Show button to hide $\Delta A^{\prime} B C$.
- Name the button "Reflect triangle $A B C$ over the line containing $B$ and $C$."
- Measure all segment lengths and all angles of the quadrilateral.
- Create Hide/Show action buttons for the constructed triangles and the measurements.

- Click on the Rotate a Triangle tab.
- Construct $\triangle A B C$.
- Construct midpoint $D$ of $\overline{A B}$.
- Construct a Circle $P$ and a line through the center of the circle.
- Construct the points of intersection, $Q$ and $R$ of the line and the circle.
- Construct point $S$ on Circle $P$.
- Select $\angle Q P S$ then from the Transform menu choose Mark Angle.
- Use the rotation by marked angle technique discussed in Unit 2 to rotate $\triangle A B C$ by the measure of $\angle Q P S$.
- Create a Move button to move the angle of rotation to $180^{\circ}$.
- Select point S.
- Select point $R$.
- From the Edit menu, choose Action Button then Movement.
- Name the button "Rotate triangle $A B C 180$ degrees about the midpoint of segment $A B$."
- Hide Circle $P$ points $Q$ and $R$ and $\overleftrightarrow{Q R}$.
- Create another Move button to move $S$ back to point $Q$. Name it "Reset."
- Draw $\overline{C^{\prime} C}$.
- Measure all segment lengths and all angles of the quadrilateral.
- Create Hide/Show action buttons for the measurements.

- Click on the Truncate a Triangle Vertex tab.
- Construct $\sqcup A B C$ using line width as dashed.
- Construct $D$ on $\overline{A B}$.
- Construct a line parallel to $\overline{A C}$ through $D$.
- Construct $E$, the intersection of $\overline{B C}$ and the parallel line.
- Hide the parallel line.
- Draw $\overline{A D}$, select $\overline{A D}$ and switch line width to thick.
- Draw $\overline{D E}$ and $\overline{E C}$
- Switch $\overline{A C}$ to line width thick.
- Measure all segment lengths and all angles of the quadrilateral.
- Calculate the ratios of the lengths of the non-parallel sides of the trapezoid to their corresponding lengths of the sides of the triangle.
- Calculate the ratios of the lengths of the sides of the smaller triangle to the corresponding lengths of the sides of the larger triangle.
- $\operatorname{Drag} A, B$, or $C$ to change the triangle.
- Drag $D$ to change the trapezoid.


## Part C

Use the quadrilateral that you selected in the Unit 4 - Informal Logic/Deductive Reasoning, Alternate Definitions activity. Use two of the properties to construct the quadrilateral with Sketchpad. Drag a vertex of one of the figures and it should remain that figure, i.e., a rectangle remains a rectangle.

## Journal Entry

Record your observations and explanations for each of the quadrilateral investigations.

- Drag the vertices of the quadrilateral to verify the properties of the quadrilaterals.


## The Geometer's Sketchpad Unit 4 <br> Area and Perimeter

Overview: In this unit, participants create interactive sketches to explore algebraic function connections with perimeter and area.

## Objective: TExES Mathematics Competencies

II.005.A. The beginning teacher understands when a relation is a function.
II.006.G. The beginning teacher models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology. III.011.B. The beginning teacher applies formulas for perimeter, area, surface area, and volume of geometric figures and shapes (e.g., polygons, pyramids, prisms, cylinders, cones, spheres) to solve problems.
III.011.C. The beginning teacher recognizes the effects on length, area, or volume when the linear dimensions of plane figures or solids are changed.
III.012.F. The beginning teacher demonstrates an understanding of the use of appropriate software to explore attributes of geometric figures and to make and evaluate conjectures about geometric relationships.
V.019.A. The beginning teacher recognizes and uses multiple representations of a mathematical concept (e.g., a point and its coordinates, the area of a circle as a quadratic function of the radius, probability as the ratio of two areas, area of a plane region as a definite integral).
V.019.E. The beginning teacher understands the use of visual media, such as graphs, tables, diagrams, and animations, to communicate mathematical information.
VI.020.D. The beginning teacher understands how learning may be enhanced through the use of manipulatives, technology, and other tools.

## Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
e.1.A. The student finds area of regular polygons and composite figures.

Materials: The Geometer's Sketchpad program, sample sketches: Unit 4-Sample Sketch 1

## Procedures:

Participants use their knowledge of The Geometer's Sketchpad to create a multi-page sketch that allows their students to make algebraic connections using the geometric concepts of area and perimeter. Participants should be able to apply their knowledge gained in previous The Geometer's Sketchpad training activities and require less guidance in the creation of this sketch. Use the Unit 4-Sample Sketch $l$ to model a sample final product for participants.

Create a multi-page sketch that will allow students to make algebraic connections using the geometric concepts of area and perimeter. Apply your knowledge gained in previous The Geometer's Sketchpad training activities to create this sketch. Use the Unit 4-Sample Sketch 1 as a guide.

- Start The Geometer's Sketchpad.
- Open a new sketch.
- Use the Document Options command to create a four page document with page titles; Introduction, Perimeter vs. Length of Long Side, Area vs. Length of Long Side, Area vs. Perimeter.
- Create Link buttons for each page.
- Create a set of instructions for the activity. (These can be modified later.)

- Click on the Perimeter vs. Length of Long Side tab.

To construct a rectangle with $\frac{\text { length of short side }}{\text { length of long side }}=\frac{1}{2}$ :

- From the Graph menu choose Grid Form, then select Rectangular.
- Construct $\overline{A D}$.
- Construct midpoint $M$ of $\overline{A D}$.
- Construct a line perpendicular to $\overline{A D}$ through $A$ and a line perpendicular to $\overline{A D}$ through $D$.
- Construct a circle centered at $A$ with radius $A M$.
- Use the Arrow tool to construct the intersection, $B$, of the circle and the perpendicular line.
- Construct a line parallel to $\overline{A D}$ through B.
- Construct the intersection, $C$, of the parallel line and the perpendicular line through $D$.
- Draw segments $\overline{A B}, \overline{B C}$, and $\overline{C D}$.


To animate the rectangle based on $\overline{A D}$ :

- Draw $\overline{A L}$ beginning at $A$ and containing $D$ longer than $\overline{A D}$.
- Select $D$ and $\overline{A L}$.
- From the Edit menu, choose Merge Point to Segment.
- Select $D$.
- From the Edit menu, choose Action Button, then select Animation.
- Make sure the message states: Point D bidirectionally on segment at medium speed.
- Click OK.
- Label the button "Change $A D$."


To generate the plot of a function Perimeter of rectangle $A B C D$ vs. $A D$ :

- Hide all construction lines and points except rectangle $A B C D$ and its vertices.
- Construct rectangle interior $A B C D$.
- Measure $A D$.
- Select the rectangle interior and measure the perimeter of $A B C D$.
- Select the rectangle interior and measure the area of $A B C D$.
- Select the measurement for $A D$ then the measurement for the Perimeter of $A B C D$.
- From the Graph menu, choose Plot As $(x, y)$.
- Label the plotted point ( $A D$, Perimeter $A B C D)$.
- From the Display menu, choose Trace Plotted Point.
- Drag the unit points on the $x$ and $y$ axes closer to origin to see more of the coordinate system.
- Add instructions.
- Create a link button back to the introduction page.
- From the Edit menu, choose Select All.
- From the Edit menu, choose Copy.
- Determine the function rule and plot it.

- Click on the Area vs. Length of Long Side tab.
- From the Edit menu, choose Paste.
- Delete the point ( $A D$, Perimeter $A B C D)$.
- Select the measurement for $I J$ then the measurement for the Area of $I K L J$.
- From the Graph menu, choose Plot As ( $x, y$ ).
- Label the plotted point (IJ, Area $I K L J$ ).
- From the Display menu, choose Trace Plotted Point.
- Modify the instructions to address this sketch.
- Create a link button back to the introduction page.
- From the Edit menu, choose Select All.
- From the Edit menu, choose Copy.
- Determine the function rule and plot it.

- Click on the Area vs. Perimeter tab.
- From the Edit menu, choose Paste.
- Delete the point (IJ, Area IKLJ).
- Select the measurement for Perimeter MNOP then the measurement for the Area MNOP.
- From the Graph menu, choose Plot As $(x, y)$.
- Label the plotted point (Perimeter MNOP, Area MNOP).
- From the Display menu, choose Trace Plotted Point.
- Modify the instructions to address this sketch.
- Create a link button back to the introduction page.
- Determine the function rule and plot it.


## Journal Entry

Record your observations and explanations for the area investigations.

- Explain why the perimeter function is linear, and the area functions are quadratic in terms of the geometry of the figures.


## The Geometer's Sketchpad Unit 5 <br> Pythagorean Theorem

Overview: In this unit, participants create interactive sketches to investigate the Pythagorean Theorem and explore sketches that are included with The Geometer's Sketchpad program.

## Objective: TExES Mathematics Competencies

II.006.G. The beginning teacher models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology. III.012.E. The beginning teacher describes and justifies geometric constructions made using compass and straightedge, reflection devices, and other appropriate technologies. III.012.F. The beginning teacher demonstrates an understanding of the use of appropriate software to explore attributes of geometric figures and to make and evaluate conjectures about geometric relationships.
III.014.C. The beginning teacher uses transformations to explore and describe reflectional, rotational, and transformational symmetry.
VI.020.D. The beginning teacher understands how learning may be enhanced through the use of manipulatives, technology, and other tools.

## Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
b.3.A. The student determines if the converse of a conditional statement is true or false.
e.1.C. The student develops, extends, and uses the Pythagorean Theorem.
e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.

Materials: The Geometer's Sketchpad program, sample sketches: Unit 5 - Sample Sketch 1, The Geometer's Sketchpad sketch: Pythagoras.

## Procedures:

## Part A

Participants use their knowledge of The Geometer's Sketchpad to create a sketch that allows their students to investigate the Pythagorean Theorem and its converse. Participants should be able to apply their knowledge gained in previous The Geometer's Sketchpad training activities and require less guidance in the creation of this sketch. Use the Unit 5-Sample Sketch 1 to model a sample final product for participants.

## Part B

Participants explore some of the sample sketches that come with the The Geometer's Sketchpad software.

Apply your knowledge gained in previous The Geometer's Sketchpad training activities to create this sketch. Use the Unit 5-Sample Sketch 1 as a guide.


- Start The Geometer 's Sketchpad.
- Open a new sketch.
- From the Graph menu, choose Show Grid.
- From the Graph menu, choose Snap Points.
- Draw right $\triangle A B C$.
- Use rotations to construct squares from the side lengths of the squares:
- Double-click on $C$ to mark $C$ as the center of rotation.
- Select $a$ and $B$.
- From the Transform menu, choose Rotate, type 90 for the angle measure, and click Rotate.
Remember that positive angles are counter-clockwise rotations.
- Repeat this process until you have constructed a square from each side of the triangle.

- Construct the square interiors.
- Measure $\angle A C B$.
- Measure lengths $a, b, c$.
- Calculate $a^{2}+b^{2}$ and $c^{2}$.
- Measure the areas of the three squares.
- Calculate the sum of areas of the two smaller squares.
- Create instructions.
- Hide grid.


## Journal Entry

Record your observations and explanations for the Pythagorean Theorem investigation.

- Explain what occurs when you drag $B$ vertically or drag $A$ horizontally to change the dimensions of the right triangle.
- Explain what occurs when you drag $B$ horizontally to change the measure of $\angle A B C$.


## Part B

Explore some of the sample sketches that come with The Geometer's Sketchpad software.


- Start The Geometer's Sketchpad.
- Click File - Open.
- Select Local Disk (C:).
- Double click the Program Files folder.
- Double click the Sketchpad folder.
- Double click the Samples folder.
- Double click the Sketchpad folder.
- Double click the Sketches folder.
- Double click the Geometry folder.
- Double click Pythagoras.gsp


Explore these sketches.

## Journal Entry

Record your observations and explanations for the Pythagorean Theorem investigation.

- Choose one of the sample Pythagoras sketches and explain how it proves the Pythagorean Theorem.


## The Geometer's Sketchpad Unit 6 Polygons

Overview: In this unit, participants create activity pages for a The Geometer's Sketchpad Sketch and create a sketch that will allow students to investigate the sum of the exterior angles of a polygon.

## Objective: TExES Mathematics Competencies

III.012.E. The beginning teacher describes and justifies geometric constructions made using compass and straightedge, reflection devices, and other appropriate technologies. III.012.F. The beginning teacher demonstrates an understanding of the use of appropriate software to explore attributes of geometric figures and to make and evaluate conjectures about geometric relationships.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
V.019.E. The beginning teacher understands the use of visual media, such as graphs, tables, diagrams, and animations, to communicate mathematical information.
VI.020.D. The beginning teacher understands how learning may be enhanced through the use of manipulatives, technology, and other tools.

## Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.

Materials: The Geometer's Sketchpad program, sample sketches: Unit 6 - Sample Sketch 1, Unit 6 Sample Sketch 2.

## Procedures:

## Part A

Participants investigate Unit 6-Sample Sketch 1 and respond to the journal entry prompts at the end of Part A.

## Part B

Participants use their knowledge of Sketchpad to create a sketch that allows their students to investigate the sum of the exterior angles of a regular polygon. Participants should be able to apply their knowledge gained in previous Sketchpad training activities and require less guidance in the creation of this sketch. Use the Unit 6-Sample Sketch 2 to model a sample final product for participants.

## Part A



## Journal Entry

Record your observations and explanations for the Pythagorean Theorem investigation.

- Describe what happens to the area of the inscribed polygon as the number of sides of the polygon increase. Explain.
- Investigate the angle measures, segment lengths, perimeters, and areas of the inscribed polygons as the number of sides increases. Describe the relationships in any algebraic rules you develop.


## Part B

Create a sketch that will allow students to investigate the sum of the exterior angles of a regular polygon. Use the Unit 6-Sample Sketch 2 as a guide.


- Start The Geometer's Sketchpad.
- Open a new sketch.
- Construct circle $O$.
- Rename the sizing point Z.
- Construct $A$ on circle $O$.
- Double click $O$ to mark it as a center of rotation.
- Select $A$ and rotate it $120^{\circ}$ about $O$.
- Rename $A^{\prime}$ to $B$.
- Select $B$ and rotate it $120^{\circ}$ about $O$.
- Rename $B^{\prime}$ to $C$.
- Construct $\overrightarrow{A B}, \overrightarrow{B C}$, and $\overrightarrow{C A}$.
- Construct $F$ on $\overrightarrow{C A}, G$ on $\overrightarrow{A B}$, and $H$ on $\overrightarrow{B C}$, all outside the circle.
- Measure $\angle F A B, \angle G B C$, and $\angle H C A$.
- Calculate the sum of the three angle measures.
- Draw $\overline{O L}$, with L outside the circle.
- Merge $Z$ onto $\overline{O L}$.
- Create an animation button to animate $Z$ bidirectionally on $\overline{O L}$.
- Name the button Change Circle Radius.
- Hide the segment portion of $\overline{O L}$ and endpoint $L$.
- Rename Z to Drag to Change Circle Radius.
- Draw any inscribed polygon of your choice and repeat this process.


## Journal Entry

Record your observations and explanations for the investigation.

## The Geometer's Sketchpad Unit 7 Fractals

Overview: In this unit, participants construct the Sierpinski Triangle and investigate the relationships which emerge from the construction.

## Objective: TExES Mathematics Competencies

II.008.A. The beginning teacher recognizes and translates among various representations (e.g., written, numerical, tabular, graphical, algebraic) of exponential and logarithmic functions.
III.012.D. The beginning teacher uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
III.012.E. The beginning teacher describes and justifies geometric constructions made using compass and straightedge, reflection devices, and other appropriate technologies. III.012.F. The beginning teacher demonstrates an understanding of the use of appropriate software to explore attributes of geometric figures and to make and evaluate conjectures about geometric relationships.
V.019.E. The beginning teacher understands the use of visual media , such as graphs, tables, diagrams, and animations, to communicate mathematical information.
VI.020.D. The beginning teacher understands how learning may be enhanced through the use of manipulatives, technology, and other tools.

## Geometry TEKS

b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
c.2. The student uses properties of transformations and their compositions to make connections between mathematics and the real world in applications such as tessellations or fractals.
e.3.A. The student uses congruence transformations to make conjectures and justify properties of geometric figures.
f.1. The student uses similarity properties and transformations to explore and justify conjectures about geometric figures.
f.2. The student uses ratios to solve problems involving similar figures.

Materials: The Geometer's Sketchpad program, sample sketches: Unit 7 - Sample Sketch 1, Sketchpad sketch, Fractal Gallery

## Procedures:

## Part A

Explain to participants that a fractal is an object or quantity, which displays self-similarity on all scales. The logarithmic spiral constructed in the Similarity Unit is an example of a fractal, as is the Sierpinski Triangle that will be constructed in this activity.

Participants will use their knowledge of The Geometer's Sketchpad to create a sketch to investigate a fractal called the Sierpinski Triangle. Use the Unit 7 - Sample Sketch 1 to model a sample final product for participants.

Participants should answer one or more of the following questions following the construction of the Sierpinski Triangle sketch.

The figures below illustrate Stages $1-3$ of the Sierpinski Triangle.


Stage 1


Stage 2


Stage 3

Use the sketch to determine the following functional relationships. Assume that the length of a side in the stage 0 triangle is 1 unit.

The total number of non-overlapping triangles versus the stage number.

$$
f(n)=3^{n-1}
$$

1. The side length of the smallest triangle in a stage versus the stage number.

$$
f(n)=\left(\frac{1}{2}\right)^{n-1}
$$

The total number of sides of non-overlapping triangles versus the stage number.

$$
f(n)=3^{n}
$$

2. The sum of the lengths of all segments versus the stage number.

$$
3+\frac{9}{2}+\frac{27}{4}+\ldots
$$

The perimeter of the Sierpinski Triangle at the nth stage, as n approaches infinity, approaches infinity.
3. The area of the smallest triangle in a stage versus the stage number.
$\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{16}, \frac{\sqrt{3}}{64}, \ldots$
The area of the Sierpinski Triangle at the nth stage, as n approaches infinity, approaches zero.

## Part B

Participants explore the Fractal Gallery sketch that is included in The Geometer's Sketchpad software. Participants respond to the journal entry prompts at the end of Part B.

## Part A

Create a sketch to investigate the properties of the Sierpinski Triangle fractal.


- Start The Geometer 's Sketchpad.
- Open a new sketch.
- Use the segment tool to construct $\triangle A B C$.
- Construct midpoints $D, E$, and $F$.
- $\quad$ Select $A, B$, and $C$.
- From the Transform menu choose Iterate.
- Click on $F, B$, then $D$ to map
$A \Rightarrow F, B \Rightarrow B, C \Rightarrow D$. (Don't click
Iterate yet.)

- Click on Structure and select Add New Map.
- Click on $E, D$, then $C$ to map $A \Rightarrow E, B \Rightarrow D, C \Rightarrow C$.
- Click on Structure and select Add New Map.
- Click on $A, F$, then $E$ to map $A \Rightarrow A, B \Rightarrow F, C \Rightarrow E$.
- Click Iterate.


- With your completed iteration selected you can increase or decrease the number of displayed iterations by pressing the + or - key on the keyboard.
- If you could iterate an infinite number of times, the result would be a Sierpinski Triangle.

The figures below illustrate Stages $1-3$ of the Sierpinski Triangle.


Stage 1


Stage 2


Stage 3

Use the sketch to determine the following functional relationships. Assume that the length of a side in the stage 1 triangle is 1 unit.

- The total number of non-overlapping triangles versus the stage number
- The side length of the smallest triangle in a stage versus the stage number
- The total number of sides of non-overlapping triangles versus the stage number
- The sum of the lengths of all segments versus the stage number
- The area of the smallest triangle in a stage versus the stage number

Part B

Explore the Fractal Gallery sketch that is included in The Geometer's Sketchpad software.


- Start The Geometer's Sketchpad.
- Click File - Open.
- Select Local Disk (C:).
- Double click the Program Files folder.
- Double click the Sketchpad folder.
- Double click the Samples folder.
- Double click the Sketchpad folder.
- Double click the Sketches folder.
- Double click the Geometry folder.
- Double click Fractal Gallery.gsp


## Journal Entry

Record your observations and explanations for the Pythagorean Theorem investigation.

- Choose one of the sample Fractal Gallery sketches and explain how it relates to the concepts investigated in Part A.


## References and Additional Resources

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## Geometry Unit

## The Texas Education Agency and the Texas Higher Education Coordinating Board Geometry Module Pre-/Post-Test

1. Triangle $S T U$ is rotated $180^{\circ}$ clockwise to form image $\Delta S^{\prime} T^{\prime} U^{\prime}$. Determine the center of rotation.

2. In the figure shown below, $m \angle C B A=50^{\circ}$ and $\triangle A B E$ is equilateral. (The figure is not drawn to scale.) Which of the following is NOT a valid conclusion for the given figure? Explain your answer.

I $\quad \overline{A E}$ bisects $\angle D A B$
II $\quad \overline{A E}$ is a median of $\triangle D A B$
III $m \angle D=30^{\circ}$
IV $\quad D E=A E$

3. The diagonals in a quadrilateral are perpendicular to each other and bisect the vertex angles of the quadrilateral. Circle all of the figures below that always have these properties.

| I | Rectangle |
| :--- | :--- |
| II | Square |
| III | Rhombus |
| IV | Parallelogram |
| V | Kite |
| VI | Isosceles Trapezoid |

4. Write a true conditional statement. Write its inverse, converse, and contrapositive. Determine whether each of these statements is true or false. Give a counterexample for each false statement.
5. In the figure shown, $\overline{A B} \square \overline{E C}, \overline{A D} \perp \overline{E C}, A B=4 \mathrm{~cm}, E C=10 \mathrm{~cm}$, and $A F=D F=3$ cm (The figure is not drawn to scale.) What is the area, in square centimeters, of pentagon $A B C D E$ ?

6. The city of Houston is building a fish pond in the middle of a popular park. The figure, not drawn to scale, represents the dimensions of the pond. The figure has one line of symmetry. A bridge will be built from $F$ to $G$. What will be the length of the bridge?

7. Circle $C$ is shown below with inscribed $\triangle A B D, m \angle A B C=15^{\circ}$, and $m \angle C B D=31^{\circ}$.


Find the measure of $\angle A C D$.
8. In the figure below, $\triangle J K M$ is a right triangle with altitude $\overline{M L}$ to the hypotenuse $\overline{K J}, m \angle K=32^{\circ}$, and $K M=6 \mathrm{~cm}$.

a) Name three pairs of similar triangles in the figure.
b) Find $M L$ and $L J$. Round your answers to the nearest tenth.
9. The two figures shown below represent the same 3-dimensional figure. The left is a perspective drawing. The right is the top view with numbers indicating how many cubes are on each stack.


Sketch a top view for the figure below, indicating how many cubes are on each stack.

10. Describe the characteristics of a surface of:
a) zero Gaussian curvature.
b) positive Gaussian curvature.
c) negative Gaussian curvature.
11. A student identifies the figure below as a rhombus but is not able to identify any of its properties. According to the van Hiele model of geometric thought, at what level is the student operating? Explain.

12. What are the advantages of using a dynamic geometry software package to teach geometry?

## The Texas Education Agency and the Texas Higher Education Coordinating Board Geometry Module Pre-/Post-Test Solutions

1. Triangle $S T U$ is rotated $180^{\circ}$ clockwise to form image $\Delta S^{\prime} T^{\prime} U^{\prime}$. Determine the center of rotation.


The point $(-1,-1)$ is the center of rotation. In general, the center of rotation can be found by finding the point of concurrency of the perpendicular bisectors of the segments connecting each pre-image vertex with its corresponding image. In the special case where the rotation is $180^{\circ}$ (clockwise or counter-clockwise) the segments connecting each vertex's pre-image with its image are concurrent at the midpoint of each segment (as shown above).
2. In the figure shown below, $m \angle C B A=50^{\circ}$ and $\triangle A B E$ is equilateral. (The figure is not drawn to scale.) Which of the following is NOT a valid conclusion for the given figure? Explain your answer.

I $\overline{A E}$ bisects $\angle D A B$
II $\overline{A E}$ is a median of $\triangle D A B$
III $m \angle D=30^{\circ}$
IV $D E=A E$


Selection I is not a valid conclusion. Since $\angle D A E, \angle E A B$, and $\angle B A C$ form a straight line, then $m \angle D A E+m \angle E A B+m \angle B A C=180^{\circ}$. Since $\triangle A B E$ is equilateral, then $m \angle E A B=60^{\circ}$. From the drawing, $m \angle B A C=90^{\circ}$. Therefore, $m \angle D A E+60^{\circ}+90^{\circ}=180^{\circ}$ and $m \angle D A E=30^{\circ}$. Since $m \angle E A B \neq m \angle D A E, \overline{A E}$ does not bisect $\angle D A B$.
3. The diagonals in a quadrilateral are perpendicular to each other and bisect the vertex angles of the quadrilateral. Circle all of the figures below that always have these properties.

I Rectangle
II Square
III Rhombus
IV Parallelogram
V Kite
VI Isosceles Trapezoid
Selections II and III always have the properties that the diagonals are perpendicular to each other and bisect the vertex angles. The diagonals of rectangles and parallelograms bisect the vertex angles, but they are not necessarily perpendicular to each other. The diagonals of a kite are perpendicular to each other, but they do not bisect the vertex angles. Isosceles trapezoids have neither of these properties.
4. Write a true conditional statement. Write its inverse, converse, and contrapositive. Determine whether each of these statements is true or false. Give a counterexample for each false statement.

Answers will vary. One example of a true conditional statement is "If I am visiting Rice University then I am in Houston, Texas." The inverse of this statement, "If I am not visiting Rice University then I am not in Houston, Texas" is false. As a counterexample, I can be at Reliant Stadium instead of Rice University and still be in Houston. The converse of the original statement, "If I am in Houston, Texas, then I am visiting Rice University," is also false. "I am in Houston, Texas and visiting Reliant Stadium," is a counterexample. The contrapositive of the statement, "If I am not visiting Houston, Texas then I am not visiting Rice University," is a true statement.
5. In the figure shown, $\overline{A B} \square \overline{E C}, \overline{A D} \perp \overline{E C}, A B=4 \mathrm{~cm}, E C=10 \mathrm{~cm}$, and $A F=D F=3$ cm (The figure is not drawn to scale.) What is the area, in square centimeters, of pentagon $A B C D E$ ?


The area of pentagon $A B C D E$ is $36 \mathrm{~cm}^{2}$.
The pentagon is a composite figure consisting of a trapezoid and a triangle.
Area of trapezoid $A B C E=\frac{1}{2} h\left(b_{1}+b_{2}\right)$

$$
=\frac{1}{2} \cdot 3 \mathrm{~cm} \cdot(4 \mathrm{~cm}+10 \mathrm{~cm})=\frac{1}{2} \cdot(3 \mathrm{~cm}) \cdot(14 \mathrm{~cm})=21 \mathrm{~cm}^{2}
$$

Area of $\triangle E D C=\frac{1}{2} \cdot(10 \mathrm{~cm}) \cdot(3 \mathrm{~cm})=15 \mathrm{~cm}^{2}$
Therefore the area of pentagon $A B C D E=21 \mathrm{~cm}^{2}+15 \mathrm{~cm}^{2}=36 \mathrm{~cm}^{2}$.
6. The city of Houston is building a fish pond in the middle of a popular park. The figure, not drawn to scale, represents the dimensions of the pond. The figure has one line of symmetry. A bridge will be built from $F$ to $G$. What will be the length of the bridge?


The bridge will be 24 yards long. Let $C$ be the intersection of $\overline{F G}$ and the line of symmetry. By the Pythagorean Theorem $(C G)^{2}+16^{2}=20^{2}$. Therefore $C G=12$ yards. Since the pond is symmetrical, then $F C=12$ yards. Therefore $F G$, the length of the bridge, is 24 yards.
7. Circle $C$ is shown below with inscribed $\triangle A B D, m \angle A B C=15^{\circ}$, and $m \angle C B D=31^{\circ}$. Find the measure of $\angle A C D$.

$m \angle A C D=92^{\circ}$. The measure of an inscribed angle is one-half the measure of its intercepted arc. Therefore $m \angle A B D=\frac{1}{2} \cdot m A D ;\left(15^{\circ}+31^{\circ}\right)=46^{\circ}=\frac{1}{2} \cdot m A D$ and $m A D=92^{\circ}$. The measure of a central angle is equal to its intercepted arc.
Therefore $m \angle A C D=m A D=92^{\circ}$ since $\angle A C D$ is a central angle with intercepted arc $A D$.
8. In the figure below, $\triangle J K M$ is a right triangle with altitude $\overline{M L}$ to the hypotenuse $\overline{K J}, m \angle K=32^{\circ}$, and $K M=6 \mathrm{~cm}$.

a) Name three pairs of similar triangles in the figure.
$\Delta K M J \square \triangle K L M$
$\triangle K M J \square \triangle M L J$
$\Delta K L M \square \triangle M L J$
b) Find $M L$ and $L J$. Round your answers to the nearest tenth.
$M L=3.2 \mathrm{~cm}$ and $L J=2.0 \mathrm{~cm}$.
From triangle $\Delta K L M, \sin 32^{\circ}=\frac{M L}{6 \mathrm{~cm}}$. Therefore
$M L=6 \cdot \sin 32^{\circ}=3.1795 \mathrm{~cm} \approx 3.2 \mathrm{~cm}$ (Note: Make sure calculators are in degree mode.)
Since $\triangle K L M \approx \triangle M L J, m \angle J M L=32^{\circ}$ and $\tan 32^{\circ}=\frac{L J}{M L}=\frac{L J}{3.1795 \mathrm{~cm}}$.
Therefore,
$L J \approx 2.0 \mathrm{~cm}$.
9. The two figures shown below represent the same 3-dimensional figure. The left is a perspective drawing. The right is the top view with numbers indicating how many cubes are on each stack.


Sketch a top view for the figure below, indicating how many cubes are on each stack.


The following is the answer:

10. Describe the characteristics of a surface of:
a) zero Gaussian curvature.

In a surface with zero Gaussian curvature, such as a plane or cylindrical surface, Euclid's first five postulates are true. Specifically the fifth postulate holds:
"Through a point not on a line, there exists exactly one line parallel to the line." Theorems whose proofs depend on this postulate are also true. For example, the sum of the measures of the angles in a triangle always equals $180^{\circ}$. A plane tangent to a surface with zero Gaussian curvature will contain a line that touches the surface at all the points on that line.
b) positive Gaussian curvature.

In a surface with positive Gaussian curvature, such as a sphere, Euclid's fifth postulate does not hold. Instead, through a point not on a line there exists no line parallel to the line. Because Euclid's fifth postulate is not true for this curvature, theorems that depend on this postulate for their proofs are not valid. As a result, for a surface with positive Gaussian curvature, the sum of the measures of the angles of a triangle is always greater than $180^{\circ}$. In addition, a plane tangent to a surface with positive Gaussian curvature will always lie completely to one side of the surface.
c) negative Gaussian curvature.

In a surface with negative Gaussian curvature, such as a pseudosphere or hyperbolic paraboloid, Euclid's fifth postulate does not hold. Instead, through a point not on a line there exists an infinite number of lines parallel to the line. Because Euclid's fifth postulate is not true for this curvature, theorems that depend on this postulate for the proofs are not valid. As a result, for a surface with negative Gaussian curvature, the sum of the measures of the angles of a triangle is always less than $180^{\circ}$. In addition a plane tangent to a surface with negative Gaussian curvature at a given point will always pass through the surface.
11. A student identifies the figure below as a rhombus but is not able to identify any of its properties. According to the van Hiele model of geometric thought, at what level is the student operating? Explain.


According to the van Hiele model of geometric thought, the student is operating at the Visual Level for this concept. In the Visual Level students are able to identify the names of geometric objects but are not yet able to specify properties.
12. What are the advantages of using a dynamic geometry software package to teach geometry?

Answers will vary. The advantages of a dynamic software package such as The Geometer's Sketchpad are many. Primarily, a dynamic geometry software package, if available in a computer laboratory setting, allows students to construct their own knowledge of geometry by allowing them to explore, make inferences, and test hypotheses. As a demonstration tool, it allows a teacher to illustrate examples more quickly and efficiently than can be done at a chalkboard.

Date:
Start time:
$\qquad$
End time: $\qquad$
Observer: $\qquad$

## I. Pre-observation interview

Discuss the lesson with the teacher, ask the following questions, and record the responses. You may need to do this interview over the phone with the teacher the night before. You may also plan ahead and send the questions to the teacher via e-mail.
A. What are the instructional goals of the activity you have planned?
B. How will the students be engaged during the lesson?
C. What student success do you expect to see take place during this activity?
D. Do you have any concerns about the activity you have planned? If so, what are they? If not, why not?
E. What should I focus on during the observation?

## Overview

## II. Observation

During the observation, make a written record of teacher and student comments and actions about the topics identified for observation during the pre-observation interview. Focus on the teacher's words and actions. Whenever possible record the teacher's exact words. Abbreviate your notes as necessary (T for teacher, G1, B1, etc. for the students). Note the time every few minutes, or when a shift or transition in the activity takes place.

As soon after the observation as possible, use your notes to write a more polished narrative. The narrative should include an accurate description of the classroom, seating arrangements, displays, etc. Draw a map of the classroom and complete the following checklist in order to provide more detailed information about its layout. The narrative should also include a list of materials used during the observed lesson. Before leaving the classroom, request copies of any worksheets that were used during your observation.
A. Physical Environment: Seating arrangement

|  | YES | NO |
| :--- | :--- | :--- |
| Students have assigned seats. |  |  |
| Desks are arranged in rows and <br> columns. |  |  |
| Desks are arranged in semi-circles. |  |  |
| Desks are arranged in clusters. |  |  |
| Tables are used rather than desks. |  |  |

B. Physical Environment: Walls

|  | YES | NO |
| :--- | :--- | :--- |
| Rules of behavior are posted. |  |  |
| Rules of math are posted (formulas, <br> process skills, problem solving <br> styles). |  |  |
| Illustrations of math concepts are <br> posted. |  |  |
| Number line is displayed. |  |  |
| Pictures are displayed. |  |  |
| Graphs or charts are displayed. |  |  |
| Motivational posters are displayed. |  |  |
| Student work is displayed. |  |  |
| Student math assignments are <br> posted. |  |  |

## Overview

C. Students

Total number of students in the classroom $\qquad$

| Ethnicity | Number of Male Students | Number of Female Students |
| :--- | :--- | :--- |
| White |  |  |
| African-American |  |  |
| Hispanic |  |  |
| Asian |  |  |
| Other |  |  |

D. Materials used during lesson

| Textbooks | Yes | No |
| :--- | :---: | :---: |
| Worksheets | Yes | No |
| Manipulatives [List which kind(s).] | Yes | No |
| Calculators | Yes | No |
| Computers | Yes | No |
| Other | Yes | No |

E. Teacher's actions during lesson

| Teacher uses an exploratory activity to <br> introduce the concept. | Yes | No |
| :--- | :---: | :---: |
| Teacher demonstrates without having <br> students participate. | Yes | No |
| Teacher has student volunteers <br> demonstrate. | Yes | No |
| Teacher leads whole class as they work <br> with demonstration materials. | Yes | No |


|  | Frequently | Sometimes | Rarely |
| :--- | :--- | :--- | :--- |
| Teacher raises questions that extend students' <br> thinking. |  |  |  |
| Teacher responds to students' questions in a <br> positive and encouraging manner. |  |  |  |
| Teacher incorporates manipulatives and technology <br> appropriately. |  |  |  |
| Teacher maintains an appropriate pace during the <br> lesson. |  |  |  |
| Teacher uses hands-on, interactive activities to <br> develop the concept (not just problems from the <br> textbook). |  |  |  |

## Overview

Teacher moves around the room to keep everyone
engaged and on track.
F. Students' actions during lesson

|  | Frequently | Sometimes | Rarely |
| :--- | :--- | :--- | :--- |
| Students are interacting with each other. |  |  |  |
| Students are working independently. |  |  |  |
| Students use a variety of materials (aside from <br> worksheets or textbook). |  |  |  |
| Students are encouraged to explain the process used to <br> reach a solution. |  |  |  |
| The majority of students are engaged in the lesson. |  |  |  |
| Students are encouraged to explore several solutions. |  |  |  |
| Students ask each other questions. |  |  |  |
| The majority of students are engaged in the <br> mathematics activity. |  |  |  |


|  | Yes | No |
| :--- | :---: | :---: |
| Students develop their own products to demonstrate <br> mastery of the concept. |  |  |
| Students are encouraged to raise original questions <br> about math and discuss these questions. |  |  |

## G. General Comments

What concept was the teacher discussing? $\qquad$

1. How long was the teacher-led portion of the lesson?

Approximate number of minutes $\qquad$
2. Describe the lesson taught.
3. How comfortable did the students appear to be with the teacher?

| Not at all comfortable | Sort of comfortable | Very comfortable |
| :---: | :---: | :---: |
|  |  |  |

4. Comments about teacher (her personality, teaching skills, rapport with students):
5. How much time was spent on student group work?

Approximate number of minutes $\qquad$
6. Describe student group work.
7. How much time was spent on individual student work?

Approximate number of minutes $\qquad$
8. Describe briefly.
9. Comments about students (their behavior, whether they were on task, understanding the lesson):
10. Please rate the quality of the teacher's classroom management.

| Poor | Adequate | Excellent |
| :---: | :---: | :---: |
|  |  |  |

11. Additional comments:

## Overview

## III. Post-observation interview

The post-interview should be done as soon after the observation as possible in order to capture data about the teacher's immediate perceptions.
A. What went particularly well during the lesson?
B. Did it differ from what you expected? If so, how?
C. If you were to teach this lesson again, what would you change?

## Overview

## Mentoring Teachers

Overview: As teachers return to the classroom with newly acquired skills and strategies from the Geometry Module, the mentoring process will allow teachers to improve classroom practice. The substance of observations and mentoring should reflect the theoretical framework of the Geometry Module that includes active, student-centered mathematical investigations, group cooperation, and alternative assessments as means to reach diverse student populations. The observations and mentoring should also allow for reflection upon the tools for learning geometry outlined in the Geometry Module: construction tools, manipulatives, and technology. These tools are used to address various learning styles, to model or represent mathematical concepts, to abstract from the manipulative representations, to construct and explore mathematical properties of geometric objects, to generate authentic data, and to progress students through the van Hiele levels.

## Objective: TExES Mathematics Competencies

VI.020.A. The beginning teacher applies research-based theories of learning mathematics to plan appropriate instructional strategies for all students.
VI.020.B. The beginning teacher understands how students differ in their approaches to learning mathematics.
VI.020.C. The beginning teacher uses students' prior mathematical knowledge to build conceptual links to new knowledge and plans instruction that builds on students' strengths and addresses students' needs.
VI.020.D. The beginning teacher understands how learning may be enhanced through the use of manipulatives, technology, and other tools. VI.020.E. The beginning teacher understands how to provide instruction along a continuum from concrete to abstract.
VI.020.F. The beginning teacher understands a variety of instructional strategies and tasks that promote students' abilities to do the mathematics described in the TEKS.
VI.020.G. The beginning teacher understands how to create a learning environment that provides all students, including English Language Learners, with opportunities to develop and improve mathematical skills and procedures.
VI.020.H. The beginning teacher understands a variety of questioning strategies to encourage mathematical discourse and to help students analyze and evaluate their mathematical thinking.
VI.020.I. The beginning teacher understands how to relate mathematics to students' lives and a variety of careers and professions.

## Overview

VI.021.A. The beginning teacher understands the purpose, characteristics, and uses of various assessments in mathematics, including formative and summative assessments.
VI.021.B. The beginning teacher understands how to select and develop assessments that are consistent with what is taught and how it is taught. VI.021.C. The beginning teacher understands how to develop a variety of assessments and scoring procedures consisting of worthwhile tasks that assess mathematical understanding, common misconceptions, and error patterns.
VI.021.D. The beginning teacher understands the relationship between assessment and instruction and knows how to evaluate assessment results to design, monitor, and modify instruction to improve mathematical learning for all students, including English Language Learners.

Background: Teachers and mentors should be well-versed in the content and pedagogy presented during the Geometry Module.

Materials: Mentoring Timeline, Classroom Observation Protocol, Observation Record: Investigations and Tools, Observation Checklist: van Hiele Levels, Observation Checklist: Cooperative Learning, Observation Checklist: Alternative Assessments, Self Report: Geometry Competencies, Self-Report: TExES Competencies

## Procedures:

Identify mentors for participants. Mentors should understand that their role is not to replicate themselves but to facilitate the professional growth of the teachers in their care. The emphasis of the process is on substance, not style. Mentoring involves more than observing and providing feedback; it requires a response to needs and situations as they happen; it includes meeting the teacher's needs with professional advice based on research rather than preference.

Mentors and participants will complete Pre-Observation Interview, Observation, and Post-Observation Interview documentation for each scheduled classroom visit/observation. The mentor and participant will meet to conduct a post-observation conference after each student classroom visit/observation.

During the pre-observation interview, the participant and mentor should review the instructional plan for TEKS appropriateness and use of pedagogical skills acquired in the Geometry Module. The teacher and mentor decide what specifically is to be observed and how this data will be collected. The teacher and mentor will select an appropriate checklist, record, or protocol form to record data during the lesson under observation.

## Overview

During the observation, the mentor should make only those observations that were discussed during the pre-observation conference. This action serves to distinguish an observation for the purpose of mentoring from an observation for the purpose of evaluation. The mentor should record what is seen or heard rather than the mentor's perceptions of, inferences about, or judgments upon what the mentor saw or heard. This objectivity provides data that accurately mirrors what took place during instructional time.

During the post-observation interview, quickly record the teacher's initial responses to the questions. The responses serve as another data source for the post-observation conference.

For the post-observation conference, prepare questions such as:

- What is another way you might have...?
- What might you see happening in your classroom if...?
- What criteria do you use to...?

These questions will help the teacher hypothesize about the results of a change in practice, analyze the effectiveness of the activity, imagine possibilities for improvement, look for patterns in teacher and student behavior, and evaluate the impact of a wellplanned activity on student achievement.

During the post-observation conference, the teacher should be allowed to speak first, sharing his or her understanding of what took place, what was successful, and what might be improved. The mentor then presents the data collected during the observation. The teacher should be allowed a few moments for quiet reflection on the data. Discussion should then take place about any discrepancies that exist between the teacher's expectations for and thoughts about the lesson and the data recorded by the mentor. When these discrepancies are identified, a professional need presents itself. They allow the teacher to determine new instructional and professional development goals to address his or her needs. They also provide opportunities for meaningful mentoring.

At the end of the year, the mentor and participant will meet to reflect again regarding the teacher's Self-Report: Geometry Competencies and the Self-Report: TExES Competencies. The teacher will establish achievable instructional and professional development goals for the following school year.

## Mentoring Timeline

| Month | Focus | Activity | Tools |
| :---: | :---: | :---: | :---: |
| August | Investigations and Tools <br> Geometry TEKS | - Pre-Observation Interview <br> - Observation <br> - Post-Observation Interview <br> - Post-Observation Conference: Establish instructional goals based on discussions between participant and mentor | - Classroom Observation Protocol <br> - Investigations and Tools <br> - van Hiele Levels |
| September | Cooperative Learning <br> Geometry TEKS | - Pre-Observation Interview <br> - Observation <br> - Post-Observation Interview <br> - Post-Observation Conference: Establish instructional goals based on discussions between participant and mentor | - Classroom Observation Protocol <br> - Cooperative Learning <br> - van Hiele Levels |
| October | Alternative <br> Assessments <br> Geometry TEKS | - Pre-Observation Interview <br> - Observation <br> - Post-Observation Interview <br> - Post-Observation Conference: Establish instructional goals based on discussions between participant and mentor | - Alternative Assessments <br> - Cooperative Learning <br> - van Hiele Levels |
| November | Investigations and Tools <br> Geometry TEKS | - Pre-Observation Interview <br> - Observation <br> - Post-Observation Interview | - Classroom Observation Protocol <br> - Investigations and Tools <br> - van Hiele Levels |


|  |  | - Post-Observation Conference: Establish instructional goals based on discussions between participant and mentor |  |
| :---: | :---: | :---: | :---: |
| January | Cooperative Learning <br> Geometry TEKS | - Pre-Observation Interview <br> - Observation <br> - Post-Observation Interview <br> - Post-Observation Conference: Establish instructional goals based on discussions between participant and mentor | - Classroom Observation Protocol <br> - Cooperative Learning <br> - van Hiele Levels |
| February | Alternative <br> Assessments <br> Geometry TEKS | - Pre-Observation Interview <br> - Observation <br> - Post-Observation Interview <br> - Post-Observation Conference: Establish instructional goals based on discussions between participant and mentor | - Alternative Assessments <br> - Cooperative Learning <br> - van Hiele Levels |
| March | Participant and Mentor will decide which of these to use as a focus: <br> Investigations and Tools <br> Cooperative <br> Learning <br> Alternative <br> Assessment <br> Geometry TEKS | Pre-Observation Interview <br> - Observation <br> - Post-Observation Interview <br> - Post-Observation Conference: Establish instructional goals based on discussions between participant and mentor | Participant and mentor will select which records to use. |
| April | Final debriefing | Generate professional development goals Identify areas for life-long learning | - Self-Report: Geometry Competencies <br> - Self-Report: TExES Competencies |

# Classroom Observation Protocol 

Observation \#: $\qquad$
Date:
Start time: $\qquad$
End time: $\qquad$
Observer: $\qquad$

## I. Pre-Observation Interview

Discuss the lesson with the teacher, ask the following questions, and record the responses. You may need to do this interview over the phone with the teacher the night before. You may also plan ahead and send the questions to the teacher via email.
A. What are the instructional goals of the activity you have planned?
B. How will the students be engaged during the lesson?
C. What student success do you expect to see take place during this activity?
D. Do you have any concerns about the activity you have planned? If so, what are they? If not, why not?
E. What should I focus on during the observation?

## II. Observation

During the observation, make a written record of teacher and student comments and actions about the topics identified for observation during the pre-observation interview. Focus on the teacher's words and actions. Whenever possible record the teacher's exact words. Abbreviate your notes as necessary (T for teacher, G1, B1 etc. for the students). Note the time every few minutes, or when a shift or transition in the activity takes place.

As soon after the observation as possible, use your notes to write a more polished narrative. The narrative should include an accurate description of the classroom, seating arrangements, displays, etc. Draw a map of the classroom and complete the following checklist in order to provide more detailed information about its layout. The narrative should also include a list of materials used during the observed lesson. Before leaving the classroom, request copies of any worksheets that were used during your observation.
A. Physical Environment: Seating arrangement

|  | YES | NO |
| :--- | :--- | :--- |
| Students have assigned seats. |  |  |
| Desks are arranged in rows and <br> columns. |  |  |
| Desks are arranged in semi-circles. |  |  |
| Desks are arranged in clusters. |  |  |
| Tables are used rather than desks. |  |  |

B. Physical Environment: Walls

|  | YES | NO |
| :--- | :--- | :--- |
| Rules of behavior are posted. |  |  |
| Rules of math are posted (formulas, <br> process skills, problem solving <br> styles). |  |  |
| Illustrations of math concepts are <br> posted. |  |  |

Overview

| Number line is displayed. |  |  |
| :--- | :--- | :--- |
| Pictures are displayed. |  |  |
| Graphs or charts are displayed. |  |  |
| Motivational posters are displayed. |  |  |
| Student work is displayed. |  |  |
| Student math assignments are <br> posted. |  |  |

C. Students

Total number of students in the classroom $\qquad$

| Ethnicity | Number of Male Students | Number of Female Students |
| :--- | :--- | :--- |
| White |  |  |
| African-American |  |  |
| Hispanic |  |  |
| Asian |  |  |
| Other |  |  |

D. Materials used during lesson

| Textbooks | Yes | No |
| :--- | :---: | :---: |
| Worksheets | Yes | No |
| Manipulatives [List which kind(s).] | Yes | No |
| Calculators | Yes | No |
| Computers | Yes | No |
| Other | Yes | No |

E. Teacher's actions during lesson

| Teacher uses an exploratory activity to <br> introduce the concept. | Yes | No |
| :--- | :---: | :---: |
| Teacher demonstrates without having <br> students participate. | Yes | No |
| Teacher has student volunteers <br> demonstrate. | Yes | No |
| Teacher leads whole class as they work <br> with demonstration materials. | Yes | No |


|  | Frequently | Sometimes | Rarely |
| :--- | :--- | :--- | :--- |
| Teacher raises questions that extend students' <br> thinking. |  |  |  |
| Teacher responds to students' questions in a <br> positive and encouraging manner. |  |  |  |
| Teacher incorporates manipulatives and technology <br> appropriately. |  |  |  |

Overview

| Teacher maintains an appropriate pace during the <br> lesson. |  |  |  |
| :--- | :--- | :--- | :--- |
| Teacher uses hands-on, interactive activities to <br> develop the concept (not just problems from the <br> textbook). |  |  |  |
| Teacher moves around the room to keep everyone <br> engaged and on track. |  |  |  |

F. Students' actions during lesson

|  | Frequently | Sometimes | Rarely |
| :--- | :--- | :--- | :--- |
| Students are interacting with each other. |  |  |  |
| Students are working independently. |  |  |  |
| Students use a variety of materials (aside from <br> worksheets or textbook). |  |  |  |
| Students are encouraged to explain the process used to <br> reach a solution. |  |  |  |
| The majority of students are engaged in the lesson. |  |  |  |
| Students are encouraged to explore several solutions. |  |  |  |
| Students ask each other questions. |  |  |  |
| The majority of students are engaged in the <br> mathematics activity. |  |  |  |


|  | Yes | No |
| :--- | :---: | :---: |
| Students develop their own products to demonstrate <br> mastery of the concept. |  |  |
| Students are encouraged to raise original questions <br> about math and discuss these questions. |  |  |

## G. General Comments

What concept was the teacher discussing? $\qquad$

1. How long was the teacher-led portion of the lesson?

Approximate number of minutes $\qquad$
2. Describe the lesson taught.
3. How comfortable did the students appear to be with the teacher?

## Not at all comfortable $\quad$ Sort of comfortable $\quad$ Very comfortable


4. Comments about teacher (her personality, teaching skills, rapport with students):
5. How much time was spent on student group work?

Approximate number of minutes $\qquad$
6. Describe student group work.
7. How much time was spent on individual student work?

Approximate number of minutes $\qquad$
8. Describe briefly.
9. Comments about students (their behavior, whether they were on task, understanding the lesson):
10. Please rate the quality of the teacher's classroom management.

| Poor | Adequate | Excellent |
| :---: | :---: | :---: |
|  |  |  |

11. Additional comments:

## III. Post-observation interview

The post-interview should be done as soon after the observation as possible in order to capture data about the teacher's immediate perceptions.
A. What went particularly well during the lesson?
B. Did it differ from what you expected? If so, how?
C. If you were to teach this lesson again, what would you change?

## Observation Record: Investigations and Tools

| Criteria | Yes | No |
| :--- | :--- | :--- |
| Students identify and apply mathematics to everyday experiences, to activities in and outside of school, with other disciplines, <br> and with other mathematical topics. |  |  |
| Student use a problem-solving model that incorporates understanding the problem, making a plan, carrying out the plan, and <br> evaluating the solution for reasonableness. |  |  |
| Students select or develop an appropriate problem-solving strategy from a variety of different types, including drawing a <br> picture, looking for a pattern, systematic guessing and checking, acting it out, making a table, working a simpler problem, or <br> working backwards to solve a problem. |  |  |
| Students select tools to solve problems. |  |  |
| Students communicate mathematical ideas using multiple representations. |  |  |
| Students evaluate the effectiveness of different representations to communicate ideas. |  |  |
| Students make conjectures from patterns or sets of examples and non-examples. |  |  |
| Students validate their conclusions using mathematical properties and relationships. |  |  |
| Students are encouraged to solve problems. |  |  |
| Students pose problems. |  |  |
| Students pose alternate solution strategies. |  |  |
| Students pose constructive questions. |  |  |
| Students apply algebraic concepts and processes. |  |  |
| Students use graphing calculators. |  |  |
| Students use interactive geometry software (e.g. The Geometer's Sketchpad, NonEuclid). |  |  |

## Observation Checklist: van Hiele Levels

| Visual Level (0) | Descriptive Level (1) | Relational Level (2) | Deductive Level (3) | Rigor (4) |
| :---: | :---: | :---: | :---: | :---: |
| - Identifies a shape. <br> - Constructs, draws, or copies a shape. <br> - Names or labels shapes and other geometric configurations appropriately. <br> - Compares and sorts shapes. <br> - Verbally describes shapes. <br> - Solves routine problems by operating on shapes. <br> - Identifies parts of a figure. | - Identifies and tests relationships. <br> - Recalls and uses appropriate vocabulary. Compares two shapes. <br> - Interprets and uses verbal description of a figure in order to draw/construct figure. Interprets verbal or symbolic statements of rules and applies them. <br> - Discovers properties of specific figures and generalizes properties. Describes a class of figures in terms of its properties. Given certain properties, identifies a shape. Connect properties between classes of figures to compare. Discovers properties of an unfamiliar class of figures. <br> - Solves geometric problems. <br> - Formulates and uses generalizations about properties of figures. | - Identifies different sets of properties that characterize a class of figures and tests that these are sufficient. Identifies minimum sets of properties that can characterize a figure. Formulates and uses a definition for a class of figures. Gives informal arguments. Gives informal deductive arguments. Gives more than one explanation to prove something and justifies these explanations. Informally recognizes difference between a statement and its converse. Identifies and uses strategies or insightful reasoning to solve problems. <br> - Recognizes the role of deductive argument and approaches problems in a deductive manner. | - Recognizes the need for undefined terms, definitions, and basic assumptions (postulates). <br> - Recognizes characteristics of a formal definition and equivalence of definitions. <br> - Proves relationships in axiomatic settings. <br> - Proves relationships between a theorem and related statements. <br> - Establishes interrelationships among networks of theorems. <br> - Compares and contrasts different proofs of theorems. <br> - Examines effects of changing an initial definition or postulate in a logical sequence. <br> - Establishes a general principle that unifies several different theorems. <br> - Creates proofs from simple sets of axioms frequently using a model to support arguments. <br> - Gives formal deductive arguments. | - Rigorously establishes theorems in different axiomatic systems. <br> - Compares axiomatic systems (e.g., Euclidean and non-Euclidean geometries); spontaneously explores how changes in axioms affect the resulting geometry. <br> Establishes consistency of a set of axioms, independence of an axiom, and equivalency of different sets of axioms; creates an axiomatic system. <br> - Invents generalized methods for solving classes of problems. <br> - Searches for the broadest context in which a mathematical theorem/principle will apply. <br> - Does in-depth study of the subject logic to develop new insights and approaches to logical inference. |

## Observation Checklist: Cooperative Learning

| Group Size <br> - 2 students <br> - 3 students <br> - 4 students <br> - Other | Purpose of Cooperative Grouping <br> - Review homework <br> - Reteach concept or procedure <br> - Developmental lesson <br> - Review lesson <br> - Test review <br> - Alternative assessment <br> - Enrichment <br> - Other | Collaborative Strategies <br> - Pair check <br> - Think-Pair-Share <br> - Group Problem Solving <br> - Jigsaw <br> - Other | Assigned Roles <br> - All students <br> - Some students <br> - None |
| :---: | :---: | :---: | :---: |
| Evidence of Group <br> Learning <br> - Positive interdependence <br> - Face-to-face promotive interaction <br> - Individual accountability <br> - Social skills <br> - Group processing | Group Success <br> - Mutual dependence <br> - Verbal interaction <br> - Interpersonal and group skills <br> - Individual accountability <br> - Incentives | Teacher <br> - Explain assignment <br> - Establish academic expectations for group <br> - Describe expected collaborative behaviors <br> - Describe group procedures <br> - Provides description of group success <br> - Monitor groups <br> - Asks for the group's answer to a student's question <br> - Affirms productive group behaviors | Closure <br> - Teacher prompts students to summarize learning <br> - Teacher ask students to address concepts/procedures not addressed in a student summary <br> - Teacher ties students' ideas together to draw closure to the learning experience |

## Observation Checklist: Alternative Assessments

| ```Type - Tasks - Project - Journal Entry - Portfolio``` | Tasks <br> - Closed <br> - Open-ended | Tools used during instruction available during assessment <br> - Yes <br> - No | Equity <br> - Wording <br> - Contexts <br> - "sense"-able <br> - relevant <br> - accessible | Opportunity for multiple sources of evidence Yes No |
| :---: | :---: | :---: | :---: | :---: |
| Student participation <br> - Actively engaged throughout the class period <br> - Actively engaged for the majority of the class period <br> - Actively engaged for about half of the class period <br> - Actively engaged for less than half of the class period <br> - Actively engaged in something other than the alternative assessment | Teacher actions <br> - Anecdotal Record <br> - Checklist <br> - Posing questions <br> - Posing problems <br> - Answering questions with a question <br> - Answering questions with an answer <br> - Redirecting student questions to other students <br> - Addressing behavioral concerns | Communication  <br> $\square$ Group <br> $\square$ Individual <br> $\square$ Model <br> $\square$ Picture <br> a Verbal <br> $\square$ Table <br> $\square$ Graph <br> $\square$ Equation | Use of Rubric <br> - Published Holistic <br> - Teacher-Made Holistic <br> - Published Analytical <br> - Teacher-Made Analytical | Student Self-Assessment <br> - Holistic Rubric <br> - Analytical Rubric <br> - Journal prompt <br> - Checklist |
| Peer assessment <br> - Yes <br> - No | Provides data to inform instruction <br> - Yes <br> - No | Content <br> - Concepts <br> - Procedures <br> - Generalizations | TEXTEAMS Geometry <br> Assessments <br> - Yes <br> - No |  |

## Self-Report: Geometry Competencies

Please rate your geometry competencies in the following areas:

| Competencies | Poor | Fair | Good | Very <br> Good | Excellent |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Knowledge of measurement <br> as a process | 0 | 1 | 2 | 3 | 4 |
| Knowledge of the results of <br> Euclidean geometry | 0 | 1 | 2 | 3 | 4 |
| Knowledge of the uses of <br> Euclidean geometry | 0 | 1 | 2 | 3 | 4 |
| Knowledge of the <br> applications of Euclidean <br> geometry | 0 | 1 | 2 | 3 | 4 |
| Knowledge of mathematical <br> reasoning | 0 | 1 | 2 | 3 | 4 |
| Knowledge of mathematical <br> problem solving | 0 | 1 | 2 | 3 | 4 |
| Knowledge of mathematical <br> connections | 0 | 1 | 2 | 3 | 4 |
| Knowledge of how children <br> learn mathematics | 0 | 1 | 2 | 3 | 4 |
| Knowledge of how to plan <br> geometry instruction | 0 | 1 | 2 | 3 | 4 |
| Knowledge of how to <br> organize geometry <br> instruction | 0 | 1 | 2 | 3 | 4 |
| Knowledge of how to <br> implement effective <br> geometry instruction | 0 | 1 | 2 | 3 | 4 |

## Self-Report: TExES Competencies

| TExES Competencies | Strongly <br> Agree | Agree | Disagree | Strongly <br> Disagree |
| :--- | :---: | :---: | :---: | :---: |
| VI.020.A. The beginning teacher applies research-based <br> theories of learning mathematics to plan appropriate <br> instructional strategies for all students. | 4 | 3 | 2 | 1 |
| VI.020.B. The beginning teacher understands how students <br> differ in their approaches to learning mathematics. | 4 | 3 | 2 | 1 |
| VI.020.C. The beginning teacher uses students' prior <br> mathematical knowledge to build conceptual links to new <br> knowledge and plans instruction that builds on students' <br> strengths and addresses students' needs. | 4 | 3 | 2 | 1 |
| II.020.D. The beginning teacher understands how learning <br> may be enhanced through the use of manipulatives, <br> technology, and other tools. | 4 | 3 | 2 | 1 |
| VI.020.E. The beginning teacher understands how to <br> provide instruction along a continuum from concrete to <br> abstract. | 4 | 3 | 2 | 1 |
| VI.020.F. The beginning teacher understands a variety of <br> instructional strategies and tasks that promote students' <br> abilities to do the mathematics described in the TEKS. | 4 | 3 | 2 | 1 |
| VI.020.G. The beginning teacher understands how to <br> create a learning environment that provides all students, <br> including English Language Learners, with opportunities <br> to develop and improve mathematical skills and <br> procedures. | 4 | 3 | 2 | 1 |
| VI.020.H. The beginning teacher understands a variety of <br> questioning strategies to encourage mathematical discourse <br> and to help students analyze and evaluate their <br> mathematical thinking. | 4 | 3 | 2 | 1 |
| VI.020.I. The beginning teacher understands how to relate <br> mathematics to students' lives and a variety of careers and <br> professions. | 4 | 3 | 2 | 1 |
| VI.021.A. The beginning teacher understands the purpose, <br> characteristics, and uses of various assessments in <br> mathematics, including formative and summative <br> assessments. | 4 | 3 | 2 | 1 |
| VI.021.B. The beginning teacher understands how to select <br> and develop assessments that are consistent with what is <br> taught and how it is taught. | 4 | 3 | 2 | 1 |
| VI.021.C. The beginning teacher understands how to <br> develop a variety of assessments and scoring procedures <br> consisting of worthwhile tasks that assess mathematical <br> understanding, common misconceptions, and error <br> patterns. | 4 | 3 | 1 |  |
| VI.021.D. The beginning teacher understands the <br> relationship between assessment and instruction and knows <br> how to evaluate assessment results to design, monitor, and | 4 | 3 | 1 |  |

modify instruction to improve mathematical learning for all students, including English Language Learners.

## References and Additional Resources

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## Texas Examination of Educator Standards Teacher Competencies for Mathematics (8-12)

## DOMAIN I-NUMBER CONCEPTS

## Competency 001

The teacher understands the real number system and its structure, operations, algorithms, and representations.

The beginning teacher:
A. Understands the concepts of place value, number base, and decimal representations of real numbers.
B. Understands the algebraic structure and properties of the real number system and its subsets (e.g., real numbers as a field, integers as an additive group).
C. Describes and analyzes properties of subsets of the real numbers (e.g., closure, identities).
D. Selects and uses appropriate representations of real numbers (e.g., fractions, decimals, percents, roots, exponents, scientific notation) for particular situations.
E. Uses a variety of models (e.g., geometric, symbolic) to represent operations, algorithms, and real numbers.
F. Uses real numbers to model and solve a variety of problems.
G. Uses deductive reasoning to simplify and justify algebraic processes.
H. Demonstrates how some problems that have no solution in the integer or rational number systems have solutions in the real number system.

## Competency 002

The teacher understands the complex number system and its structure, operations, algorithms, and representations.

The beginning teacher:
A. Demonstrates how some problems that have no solution in the real number system have solutions in the complex number system.
B. Understands the properties of complex numbers (e.g., complex conjugate, magnitude/modulus, multiplicative inverse).
C. Understands the algebraic structure of the complex number system and its subsets (e.g., complex numbers as a field, complex addition as vector addition).
D. Selects and uses appropriate representations of complex numbers (e.g., vector, ordered pair, polar, exponential) for particular situations.
E. Describes complex number operations (e.g., addition, multiplication, roots) using symbolic and geometric representations.

## Competency 003

The teacher understands number theory concepts and principles and uses numbers to model and solve problems in a variety of situations.

The beginning teacher:
A. Applies ideas from number theory (e.g., prime numbers and factorization, the Euclidean algorithm, divisibility, congruence classes, modular arithmetic, the fundamental theorem of arithmetic) to solve problems.
B. Applies number theory concepts and principles to justify and prove number relationships.
C. Compares and contrasts properties of vectors and matrices with properties of number systems (e.g., existence of inverses, non-commutative operations).
D. Uses properties of numbers (e.g., fractions, decimals, percents, ratios, proportions) to model and solve real-world problems.
E. Applies counting technique such as permutations and combinations to quantify situations and solve problems.
F. Uses estimation techniques to solve problems and judges the reasonableness of solutions.

## DOMAIN II-PATTERNS AND ALGEBRA

## Competency 004

The teacher uses patterns to model and solve problems and formulate conjectures.
The beginning teacher:
A. Recognizes and extends patterns and relationships in data presented in tables, sequences, or graphs.
B. Uses methods of recursion and iteration to model and solve problems.
C. Uses the principle of mathematical induction.
D. Analyzes the properties of sequences and series (e.g., Fibonacci, arithmetic, geometric) and uses them to solve problems involving finite and infinite processes.
E. Understands how sequences and series are applied to solve problems in the mathematics of finance (e.g., simple, compound, and continuous interest rates; annuities).
Competency 005
The teacher understands attributes of functions, relations, and their graphs.

The beginning teacher:
A. Understands when a relation is a function.
B. Identifies the mathematical domain and the range of functions and relations and determines reasonable domains for given situations.
C. Understand that a function represents a dependence of one quantity on another and can be represented in a variety of ways (e.g., concrete models, tables, graphs, diagrams, verbal descriptions, symbols).
D. Identifies and analyzes even and odd functions, one-to-one functions, inverse functions, and their graphs.
E. Applies basic transformations [e.g., $k f(x), f(x)+k, f(x-k), f(k x),|f(x)|$ to a parent function, $f$, and describes the effects on the graph of $y=f(x)$.
F. Performs operations (e.g., sum, difference, composition) on functions, finds inverse relations, and describes results symbolically and graphically.
G. Uses graphs of functions to formulate conjectures of identities [e.g., $y=x^{2}-1$ and $y=(x-1)(x+1), y=\log x^{3}$ and $y=3 \log x, y=\sin \left(x+\frac{\pi}{2}\right)$ and $\left.y=\cos x\right]$.

## Competency 006

The teacher understands linear and quadratic functions, analyzes their algebraic and graphical properties, and uses them to model and solve problems.

The beginning teacher:
A. Understands the concept of slope as a rate of change and interprets the meaning of slope and intercept in a variety of situations.
B. Writes equations of lines given various characteristics (e.g., two points, a point and slope, slope and y-intercept).
C. Applies techniques of linear and matrix algebra to represent and solve problems involving linear systems.
D. Analyzes the zeros (real and complex) of quadratic functions.
E. Makes connections between the $y=a x^{2}+b x+c$ and the $\mathrm{y}=a(x-h)^{2}+\mathrm{k}$ representations of a quadratic function and its graph.
F. Solves problems involving quadratic functions using a variety of methods (e.g., factoring, completing the square, using the quadratic formula, using a graphing calculator).
G. Models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology.

## Competency 007

The teacher understands polynomial, rational, radical, absolute value, and piecewise functions, analyzes their algebraic and graphical properties, and uses them to model and solve problems.

The beginning teacher:
A. Recognizes and translates among various representations (e.g., written, tabular, graphical, algebraic) of polynomial, rational, radical, absolute value, and piecewise functions.
B. Describes restrictions on the domains and ranges of polynomial, rational, radical, absolute value, and piecewise functions.
C. Makes and uses connections among the significant points (e.g., zeros, local extrema, points where a function is not continuous or not differentiable) of a function, the graph of a function, and the function's symbolic representation.
D. Analyzes functions in terms of vertical, horizontal, and slant asymptotes.
E. Analyzes and applies the relationship between inverse variation and rational functions.
F. Solves equations and inequalities involving polynomial, rational, radical, absolute value, and piecewise functions using a variety of methods (e.g., tables, algebraic methods, graphs, use of graphing calculator), and evaluates the reasonableness of solutions.
G. Models situations using polynomial, rational, radical, absolute value, and piecewise functions and solves problems using a variety of methods, including technology.

## Competency 008

The teacher understands exponential and logarithmic functions, analyzes their algebraic and graphical properties, and uses them to model and solve problems.

The beginning teacher:
A. Recognizes and translates among various representations (e.g., written, numerical, tabular, graphical, algebraic) of exponential and logarithmic functions.
B. Recognizes and uses connections among significant characteristics (e.g., intercepts, asymptotes) of a function involving exponential or logarithmic expressions, the graph of the function, and the function's symbolic representation.
C. Understands the relationship between exponential and logarithmic functions and uses the laws and properties of exponents and logarithms to simplify expressions and solve problems.
D. Uses a variety of representations and techniques (e.g., numerical methods, tables, graphs, analytic techniques, graphing calculators) to solve equations, inequalities, and systems involving exponential and logarithmic functions.
E. Models and solves problems involving exponential growth and decay.
F. Uses logarithmic scales (e.g., Richter, decibel) to describe phenomena and solve problems.
G. Uses exponential and logarithmic functions to model and solve problems involving the mathematics of finance. (e.g., compound interest).
H. Uses the exponential function to model situations and solve problems in which the rate of change of a quantity if proportional to the current amount of the quantity [i.e., $f^{\prime}(x)=k f(x)$ ].

## Competency 009

The teacher understands trigonometric and circular functions, analyzes their algebraic and graphical properties, and uses them to model and solve problems.

The beginning teacher:
A. Analyzes the relationship among the unit circle in the coordinate plane, circular functions, and the trigonometric functions.
B. Recognizes and translates among various representations (e.g., written, numerical, tabular, graphical, algebraic) of trigonometric functions and their inverses.
C. Recognizes and uses connections among significant properties (e.g., zeros, axes of symmetry, local extrema) and characteristics (e.g., amplitude, frequency, phase shift) of a trigonometric function, the graph of the function, and the function's symbolic representation.
D. Understands the relationship between trigonometric functions and their inverses and uses these relationships to solve problems.
E. Uses trigonometric identities to simplify expressions and solve equations.
F. Models and solves a variety of problems (e.g., analyzing periodic phenomena) using trigonometric functions.
G. Uses graphing calculators to analyzes and solve problems involving trigonometric functions.

## Competency 010

The teacher understands and solves problems using differential and integral calculus.

The beginning teacher:
A. Understands the concept of limit and the relationship between limits and continuity.
B. Relates the concept of average rate of change to the slope of the secant line and the concept of instantaneous rate of change to the slope of the tangent line.
C. Uses the first and second derivations to analyze the graph of a function (e.g., local extrema, concavity, points of inflection).
D. Understands and applies the fundamental theorem of calculus and the relationship between differentiation and integration.
E. Models and solves a variety of problems (e.g., velocity, acceleration, optimization, related rates, work, center of mass) using differential and integral calculus.
F. Analyzes how technology can be used to solve problems and illustrate concepts involving differential and integral calculus.

## DOMAIN III-GEOMETRY AND MEASUREMENT

## Competency 011

The teacher understands measurement as a process.
The beginning teacher:
A. Applies dimensional analysis to derive units and formulas in a variety of situations (e.g., rates of change of one variable with respect to another) and to find and evaluate solutions to problems.
B. Applies to formulas for perimeter, area, surface area, and volume of geometric figures and shapes (e.g., polygons, pyramids, prisms, cylinders, cones, spheres) to solve problems.
C. Recognizes the effects on length, area, or volume when the linear dimensions of plane figures or solids are changed.
D. Applies the Pythagorean theorem, proportional reasoning, and right triangle trigonometry to solve measurement problems.
E. Relates the concept of area under a curve to the limit of a Riemann sum.
F. Uses integral calculus to compute various measurements associated with curves and regions (e.g., area, arc length) in the plane and measurements associated with curves, surfaces, and regions in three-space.

## Competency 012

The teacher understands geometries, in particular Euclidean geometry, as axiomatic systems.

The beginning teacher:
A. Understands axiomatic systems and their components (e.g., undefined terms, defined terms, theorems, examples, counterexamples).
B. Uses properties of points, lines, planes, angles, lengths, and distances to solve problems.
C. Applies the properties of parallel and perpendicular lines to solve problems.
D. Uses properties of congruence and similarity to explore geometric relationships, justify conjectures, and prove theorems.
E. Describes and justifies geometric constructions made using compass and straightedge, reflection devices, and other appropriate technologies.
F. Demonstrates an understanding of the use of appropriate software to explore attributes of geometric figures and to make and evaluate conjectures about geometric relationships.
G. Compares and contrasts the axioms of Euclidean geometry with those of nonEuclidean geometry (e.g., hyperbolic and elliptic geometry).

## Competency 013

The teacher understands the results, uses, and applications of Euclidean geometry.
The beginning teacher:
A. Analyzes the properties of polygons and their components.
B. Analyzes the properties of circles and the lines that intersect them.
C. Uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two-and three-dimensional figures and shapes (e.g., relationships of sides, angles).
D. Computes the perimeter, area, and volume of figures and shapes created by subdividing and combining other figures and shapes (e.g., arc length, area of sectors).
E. Analyzes cross-sections and nets of three-dimensional shapes.
F. Uses top, front, side, and corner views of three-dimensional shapes to create complete representations and solve problems.
G. Applies properties of two- and three-dimensional shapes to solve problems across the curriculum and in everyday life.

## Competency 014

The teacher understands coordinate, transformational, and vector geometry and their connections.

The beginning teacher:
A. Identifies transformations (i.e., reflections, translations, glide-reflections, rotations, dilations) and explores their properties.
B. Uses the properties of transformations and their compositions to solve problems.
C. Uses transformations to explore and describe reflectional, rotational, and translational symmetry.
D. Applies transformations in the coordinate plane.
E. Applies concepts and properties of slope, midpoint, parallelism, perpendicularity, and distance to explore properties of geometric figures and solve problems in the coordinate plane.
F. Uses coordinate geometry to derive and explore the equations, properties, and applications of conic sections. (i.e., lines, circles, hyperbolas, ellipses, parabolas).
G. Relates geometry and algebra by representing transformations as matrices and uses this relationship to solve problems.
H. Explores the relationship between geometric and algebraic representations of vectors and uses this relationship to solve problems.

## DOMAIN IV-PROBABILITY AND STATISTICS

## Competency 015 <br> The teacher understands how to use appropriate graphical and numerical techniques to explore data, characterize patterns, and describe departures from patterns.

The beginning teacher:
A. Selects and uses an appropriate measurement scale (i.e., nominal, ordinal, interval, ratio) to answer research questions and analyzes data.
B. Organizes, displays, and interprets data in a variety of formats (e.g., tables, frequency distributions, scatter plots, stem-and-leaf plots, box-and-whisker plots, histograms, pie charts).
C. Applies concepts of center, spread, shape, and skewness to describe a data distribution.
D. Understands measures of central tendency (i.e., mean, median, mode) and dispersion (i.e., range, interquartile range, variance, standard deviation).
E. Applies linear transformations (i.e., translating, stretching, shrinking) to convert data and describes the effect of linear transformations on measures of central tendency and dispersion.
F. Analyzes connections among concepts of center and spread, data clusters and gaps, data outliers, and measures of central tendency and dispersion.
G. Supports arguments, makes predictions, and draws conclusions using summary statistics and graphs to analyze and interpret one-variable data.

## Competency 016

The teacher understands concepts and applications of probability.
The beginning teacher:
A. Understands how to explore concepts of probability through sampling, experiments, and simulations, and generates and uses probability models to represent situations.
B. Uses the concepts and principles of probability to describe the outcomes of simple and compound events.
C. Determines probabilities by constructing sample spaces to model situations.
D. Solves a variety of probability problems using combinations and permutations.
E. Solves a variety of probability problems using ratios of areas of geometric regions.
F. Calculates probabilities using the axioms of probability and related theorems and concepts such as the addition rule, multiplication rule, conditional probability, and independence.
G. Understands expected value, variance, and standard deviation of probability distributions (e.g., binomial, geometric, uniform, normal).
H. Applies concepts and properties of discrete and continuous random variables to model and solve a variety of problems involving probability and probability distributions (e.g., binomial, geometric, uniform, normal).

## Competency 017

The teacher understands the relationship among probability theory, sampling, and statistical inference, and how statistical inference is used in making and evaluating predictions.

The beginning teacher:
A. Applies knowledge of designing, conducting, analyzing, and interpreting statistical experiments to investigate real-world problems.
B. Analyzes and interprets statistical information (e.g., the results of polls and surveys) and recognize misleading as well as valid uses of statistics.
C. Understands random samples and sample statistics (e.g., the relationship between sample size and confidence intervals, biased or unbiased estimators).
D. Makes inferences about a population using binomial, normal, and geometric distributions.
E. Describes and analyzes bivariate data using various techniques (e.g., scatterplots, regression lines, outliers, residual analysis, correlation coefficients).
F. Understands how to transform nonlinear data into linear form in order to apply linear regression techniques to develop exponential, logarithmic, and power regression models.
G. Uses the law of large numbers and the central limit theorem in the process of statistical inference.
H. Estimates parameters (e.g., population mean and variance) using point estimators (e.g., sample mean and variance).
I. Understands principles of hypotheses testing.

## DOMAIN V-MATHEMATICAL PROCESSES AND PERSPECTIVES

Competency 018
The teacher understands mathematical reasoning and problem solving.
The beginning teacher:
A. Recognizes and uses multiple representations of a mathematical concept (e.g., a point and its coordinates, the area of a circle as a quadratic function of the radius, probability as the ratio of two areas, area of a plane region as a definite integral).
B. Understands how mathematics is used to model and solve problems in other disciplines (e.g., art, music, science, social science, business).
C. Uses inductive reasoning to make conjectures and uses deductive methods to evaluate the validity of conjectures.
D. Uses formal and informal reasoning to justify mathematical ideas.
E. Understands the problem-solving process (i.e., recognizing that a mathematical problem can be solved in a variety of ways, selecting an appropriate strategy, evaluating the reasonableness of a solution).
F. Evaluates how well a mathematical model represents a real-world situation.

## Competency 019

The teacher understands mathematical connections both within and outside of mathematics and how to communicate mathematical ideas and concepts.

The beginning teacher:
A. Recognizes and uses multiple representations of a mathematical concept (e.g., a point and its coordinates, the area of the circle as a quadratic function of the radius, probability as the ratio of two areas, area of a plane region as a definite integral).
B. Understands how mathematics is used to model and solve problems in other disciplines (e.g., art, music science, social science, business).
C. Translates mathematical ideas between verbal and symbolic forms.
D. Communicates mathematical ideas using a variety of representations (e.g., numeric, verbal, graphical, pictorial, symbolic, concrete).
E. Understands the use of visual media, such as graphs, tables, diagrams, and animations, to communicate mathematical information.
F. Uses appropriate mathematical terminology to express mathematical ideas.

## DOMAIN VI-MATHEMATICAL LEARNING, INSTRUCTION, AND ASSESSMENT

Competency 020
The teacher understands how children learn mathematics and plans, organizes, and implements instruction using knowledge of students, subject matter, and statewide curriculum (Texas Essential Knowledge and Skills [TEKS]).

The beginning teacher:
A. Applies research-based theories of learning mathematics to plan appropriate instructional activities for all students.
B. Understands how students differ in their approaches to learning mathematics.
C. Uses students' prior mathematical knowledge to build conceptual links to new knowledge and plans instruction that builds on students' strengths and addresses students' needs.
D. Understands how learning may be enhanced through the use of manipulatives, technology, and other tools (e.g., stop watches, scales, rulers).
E. Understands how to provide instructions along a continuum from concrete to abstract.
F. Understands a variety of instructional strategies and tasks that promote students' abilities to do the mathematics described in the TEKS.
G. Understands how to create a learning environment that provides all students, including English Language Learners, with opportunities to develop and improve mathematical skills and procedures.
H. Understands a variety of questioning strategies to encourage mathematical discourse and to help students analyze and evaluate their mathematical thinking.
I. Understands how to relate mathematics to students' lives and a variety of careers and professions.

## Competency 021

The teacher understands assessment and uses a variety of formal and informal assessment techniques to monitor and guide mathematics instruction and to evaluate student progress.

The beginning teacher:
A. Understands the purpose, characteristics, and uses of various assessments in mathematics, including formative and summative assessments.
B. Understands how to select and develop assessments that are consistent with what is taught and how it is taught.
C. Understands how to develop a variety of assessment and scoring procedures consisting of worthwhile tasks that assess mathematical understanding, common misconceptions, and error patterns.
D. Understands the relationship between assessment and instruction and knows how to evaluate assessment results to design, monitor, and modify instruction to improve mathematical learning for all students, including English Language Learner.

## Texas Essential Knowledge and Skills for Geometry

(a) Basic understandings.
(1) Foundation concepts for high school mathematics. As presented in Grades K-8, the basic understandings of number, operation, and quantitative reasoning; patterns, relationships, and algebraic thinking; geometry; measurement; and probability and statistics are essential foundations for all work in high school mathematics. Students continue to build on this foundation as they expand their understanding through other mathematical experiences.
(2) Geometric thinking and spatial reasoning. Spatial reasoning plays a critical role in geometry; shapes and figures provide powerful ways to represent mathematical situations and to express generalizations about space and spatial relationships. Students use geometric thinking to understand mathematical concepts and the relationships among them.
(3) Geometric figures and their properties. Geometry consists of the study of geometric figures of zero, one, two, and three dimensions and the relationships among them. Students study properties and relationships having to do with size, shape, location, direction, and orientation of these figures.
(4) The relationship between geometry, other mathematics, and other disciplines. Geometry can be used to model and represent many mathematical and real-world situations. Students perceive the connection between geometry and the real and mathematical worlds and use geometric ideas, relationships, and properties to solve problems.
(5) Tools for geometric thinking. Techniques for working with spatial figures and their properties are essential in understanding underlying relationships. Students use a variety
of representations (concrete, pictorial, algebraic, and coordinate), tools, and technology, including, but not limited to, powerful and accessible hand-held calculators and computers with graphing capabilities to solve meaningful problems by representing figures, transforming figures, analyzing relationships, and proving things about them.
(6) Underlying mathematical processes. Many processes underlie all content areas in mathematics. As they do mathematics, students continually use problem-solving, computation in problem-solving contexts, language and communication, connections within and outside mathematics, and reasoning, as well as multiple representations, applications and modeling, and justification and proof.
(b) Geometric structure: knowledge and skills and performance descriptions.
(1) The student understands the structure of, and relationships within, an axiomatic system. Following are performance descriptions.
(A) The student develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems.
(B) Through the historical development of geometric systems, the student recognizes that mathematics is developed for a variety of purposes.
(C) The student compares and contrasts the structures and implications of Euclidean and non-Euclidean geometries.
(2) The student analyzes geometric relationships in order to make and verify conjectures. Following are performance descriptions.
(A) The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
(B) The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
(3) The student understands the importance of logical reasoning, justification, and proof in mathematics. Following are performance descriptions.
(A) The student determines if the converse of a conditional statement is true or false.
(B) The student constructs and justifies statements about geometric figures and their properties.
(C) The student demonstrates what it means to prove mathematically that statements are true.
(D) The student uses inductive reasoning to formulate a conjecture.
(E) The student uses deductive reasoning to prove a statement.
(4) The student uses a variety of representations to describe geometric relationships and solve problems.

Following is a performance description. The student selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.
(c) Geometric patterns: knowledge and skills and performance descriptions.

The student identifies, analyzes, and describes patterns that emerge from two- and threedimensional geometric figures. Following are performance descriptions.
(1) The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
(2) The student uses properties of transformations and their compositions to make connections between mathematics and the real world in applications such as tessellations or fractals.
(3) The student identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.
(d) Dimensionality and the geometry of location: knowledge and skills and performance descriptions.
(1) The student analyzes the relationship between three-dimensional objects and related two-dimensional representations and uses these representations to solve problems. Following are performance descriptions.
(A) The student describes, and draws cross sections and other slices of threedimensional objects.
(B) The student uses nets to represent and construct three-dimensional objects.
(C) The student uses top, front, side, and corner views of three-dimensional objects to create accurate and complete representations and solve problems.
(2) The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly. Following are performance descriptions.
(A) The student uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.
(B) The student uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons.
(C) The student develops and uses formulas including distance and midpoint.
(e) Congruence and the geometry of size: knowledge and skills and performance descriptions.
(1) The student extends measurement concepts to find area, perimeter, and volume in problem situations. Following are performance descriptions.
(A) The student finds areas of regular polygons and composite figures.
(B) The student finds areas of sectors and arc lengths of circles using proportional reasoning.
(C) The student develops, extends, and uses the Pythagorean Theorem.
(D) The student finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations.
(2) The student analyzes properties and describes relationships in geometric figures. Following are performance descriptions.
(A) Based on explorations and using concrete models, the student formulates and tests conjectures about the properties of parallel and perpendicular lines.
(B) Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.
(C) Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of circles and the lines that intersect them.
(D) The student analyzes the characteristics of three-dimensional figures and their component parts.
(3) The student applies the concept of congruence to justify properties of figures and solve problems. Following are performance descriptions.
(A) The student uses congruence transformations to make conjectures and justify properties of geometric figures.
(B) The student justifies and applies triangle congruence relationships.
(f) Similarity and the geometry of shape: knowledge and skills and performance descriptions. The student applies the concepts of similarity to justify properties of figures and solve problems.
Following are performance descriptions.
(1) The student uses similarity properties and transformations to explore and justify conjectures about geometric figures.
(2) The student uses ratios to solve problems involving similar figures.
(3) In a variety of ways, the student develops, applies, and justifies triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples.
(4) The student describes the effect on perimeter, area, and volume when length, width, or height of a three-dimensional solid is changed and applies this idea in solving problems.

| $\square$ | - |  |  |  |  |  |  |  |  |
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