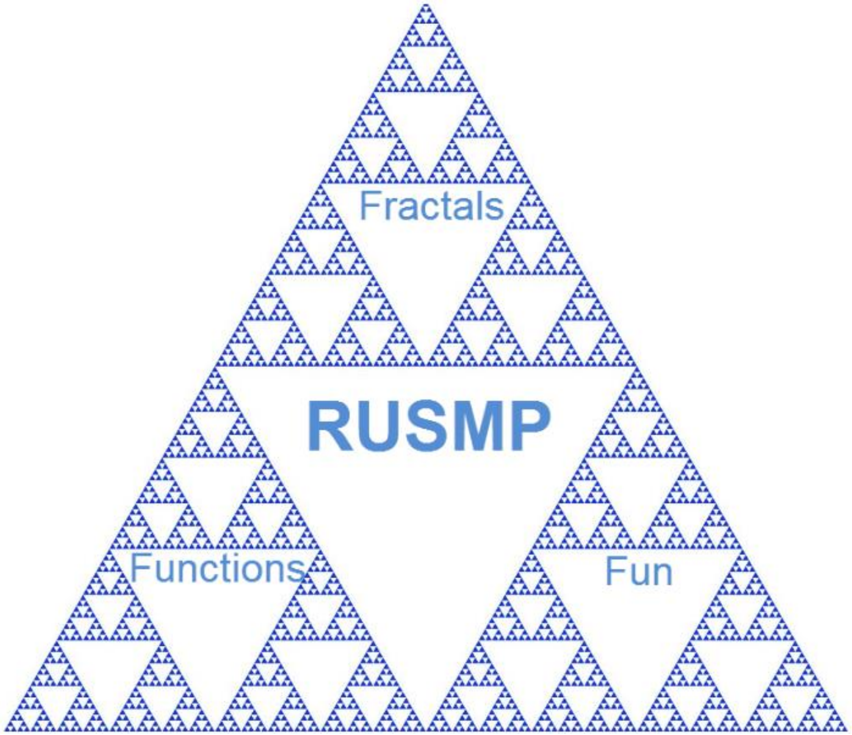
Fun with Fractals and Functions



NCTM Regional Conference
November 20, 2014
Houston, Texas

Alice Fisher
afisher@rice.edu
Director of Technology Applications & Integration

Susan Troutman
troutman@rice.edu
Director of Secondary Programs

What is a fractal?

Fractal comes from the Latin word "fractus" for *broken* or *fractured* and was coined by the mathematician Benoit Mandelbrot in 1975.

Mandelbrot showed how visual complexity can be created from simple rules.

What is a fractal?

- Fractals are formed by evaluating a function over and over again for which the output becomes the input in the next iteration. This process is called **recursion.**

Example: $$\mathbf{z_{n+1} = (z_n)^2 + C}$$

With $z_0 = C$ where C is a point in the complex plane. The Mandelbrot set is the set of values of C for which the orbit under iteration remains **bounded**.

What is a fractal?

$$z_{n+1} = (z_n)^2 + C$$

When C = 1:

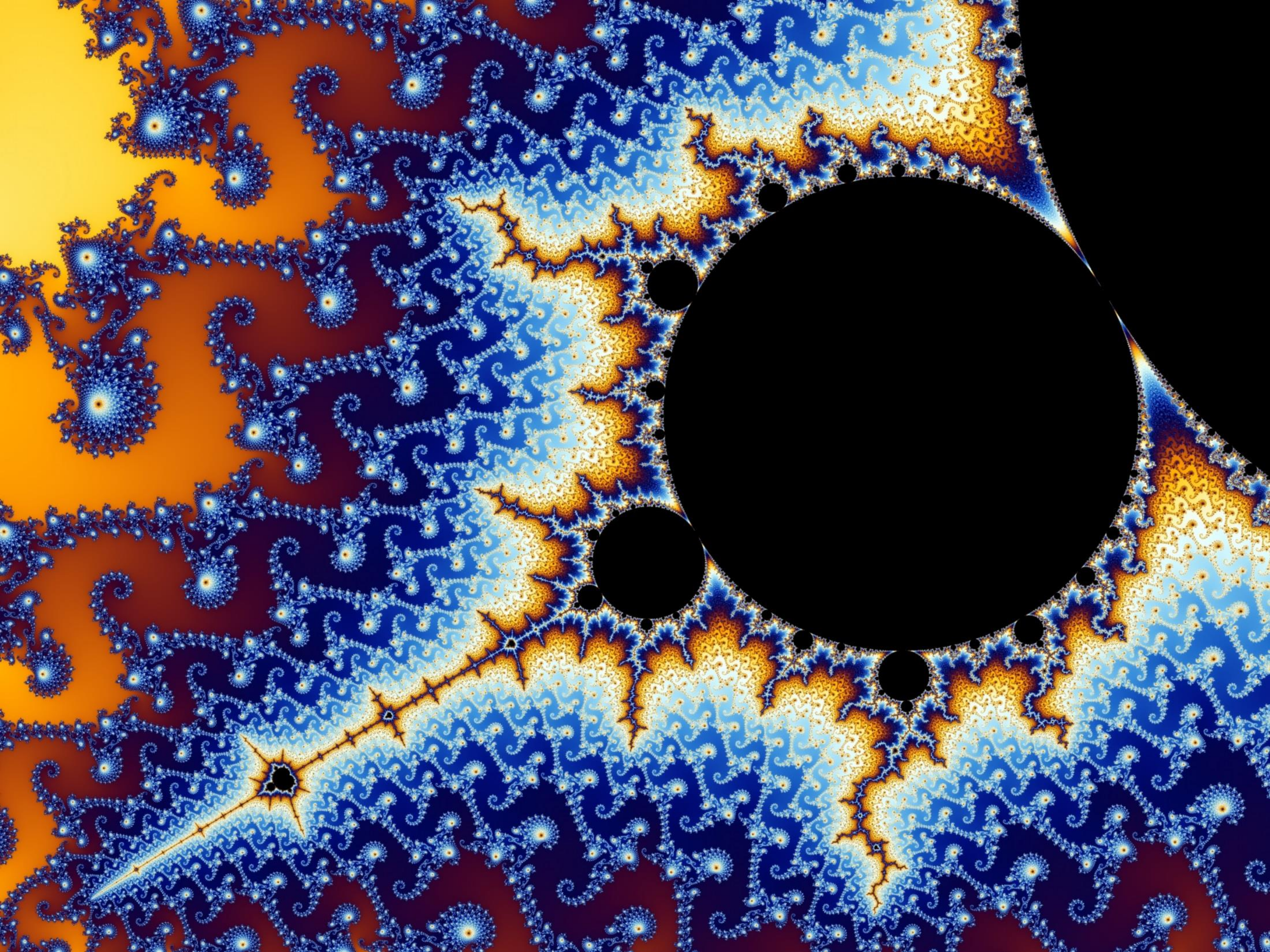
$z_0 = 1,\ z_1 = 1^2 + 1 = 2,\ \ z_2 = 2^2 + 1 = 5,\ \ z_3 = 5^2 + 1 = 26, \ldots$

Since this sequence tends to infinity, C = 1 is not in the Mandelbrot set.

When C = -1:

$z_0 = -1,\ \ z_1 = (-1)^2 + (-1) = 0,\ \ z_2 = 0^2 + (-1) = -1,$
$z_3 = (-1)^2 + (-1) = 0, \ldots$

Since this sequence is bounded, C = -1 is in the Mandelbrot set.



What is a fractal?

- When viewed closely, fractals are **self-similar**. Thus, if we zoom in on a fractal, we essentially see the same shape, but at a different scale.
- Thus, "smaller and smaller copies of a pattern are successively nested inside each other, so that the same intricate shapes appear no matter how much you zoom in to the whole."(Stephen Wolfram, 2012)

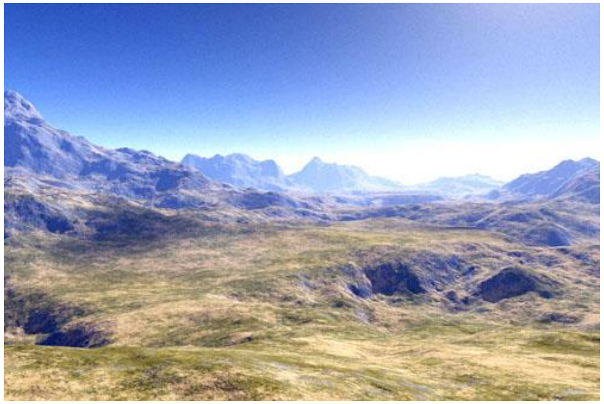
Xaos Software Demo

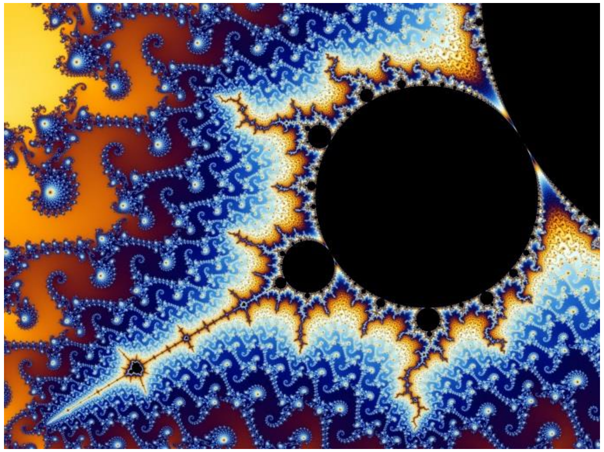
Let’s zoom in!

Where do we find fractals?

We find fractals in nature! Examples include **branching patterns** such as trees, river networks, blood vessels, mountains (below left) and **spiral patterns** such as seashells, hurricanes and galaxies (below right).

Fractals Created by Software



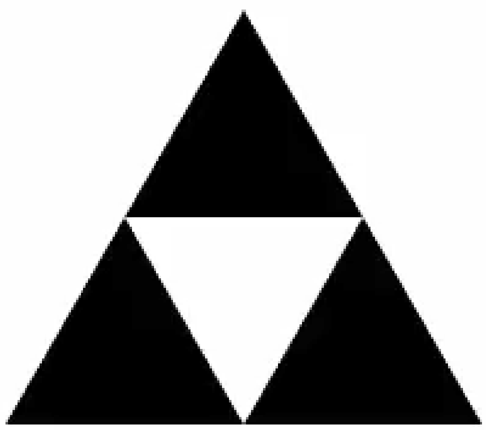


Why study fractals in the mathematics classroom?

- Fractals are relevant! They are found almost everywhere in nature.
- There is a lot of mathematics underlying fractals that is accessible to secondary school students.
- Fractals combine mathematics and art, and their beauty and diversity may engage students who are not normally “hooked” in a mathematics classroom.

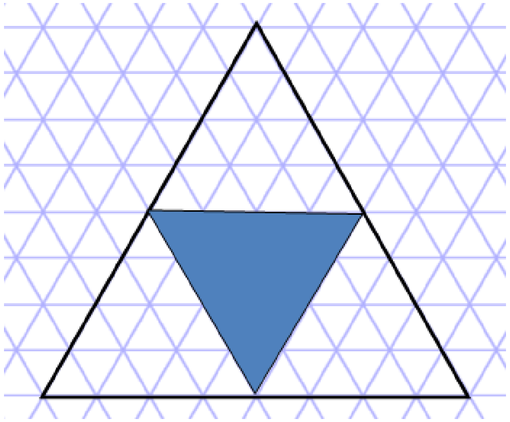
The Sierpinski Triangle

The **Sierpinski triangle** is a fractal named after the Polish mathematician, Wacław Sierpiński, who described it in 1915.

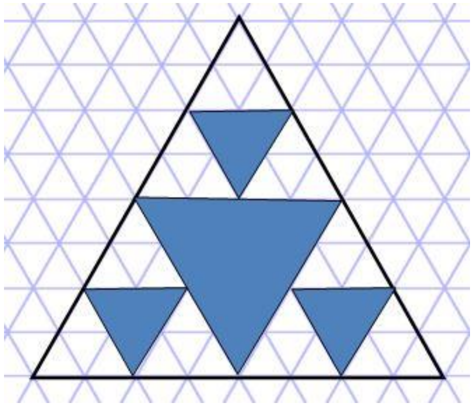


Let's create the first three iterations of the Sierpinski triangle:

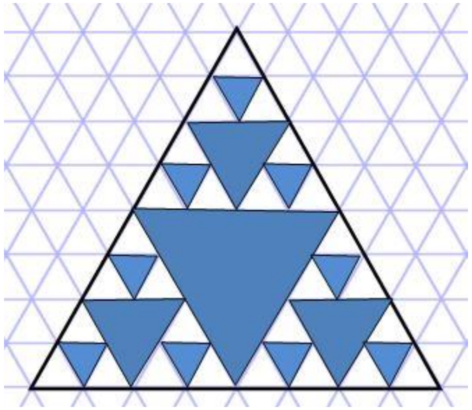
Iteration 1: Draw an equilateral triangle with side length of 8 units on triangular grid paper. Mark the midpoints of the three sides. Then connect the three midpoints and shade in the triangle that is pointing downward.



Iteration 2: Repeat the first iteration with a new triangle. Now mark the midpoints of the three sides of each of the three unshaded triangles. Connect the midpoints and shade the three triangles that are pointing downward.



<u>Iteration 3</u>: Repeat the first and second iterations with a new triangle. Now mark the midpoints of the three sides of each of the nine unshaded triangles. Connect the midpoints and shade/color the nine triangles that are pointing downward. Be creative when you shade in your triangles…use colors! Cut out the three triangles.



What do you notice?

What do you wonder?

Share with your neighbor.

We will examine **4 functions** derived from looking at **successive iterations** of the Sierpinski triangle:

1. The number of unshaded triangles within the Sierpinski triangle
2. The ratio of the area of the Sierpinski triangle that is unshaded to the area of the entire Sierpinski triangle
3. The length of the boundary of the Sierpinski triangle
4. The number of shaded triangles within the Sierpinski triangle

Introducing the Chaos Game

In 1989, the term “Chaos Game” was coined by Michael Barnsley who developed this technique.

How do you play the Chaos Game?

1. Use the triangle handout and work in pairs.
2. Pick a point inside the triangle to begin the game. This point is called the seed.
3. Each vertex of the triangle has been labeled with two numbers as follows:

   Top vertex: 1, 2
   Bottom left vertex: 3, 4
   Bottom right vertex: 5, 6.

How do you play the Chaos Game?

4. One person rolls the die and the other person plots the next point half the distance from the seed to the vertex that corresponds to the result of the rolled die.
5. Roll the die again, and then plot the next point half the distance from the last plotted point and the vertex that corresponds to the result of the rolled die.
6. Plot 5-10 points on the triangle.

What do you predict you will see after

100 iterations?

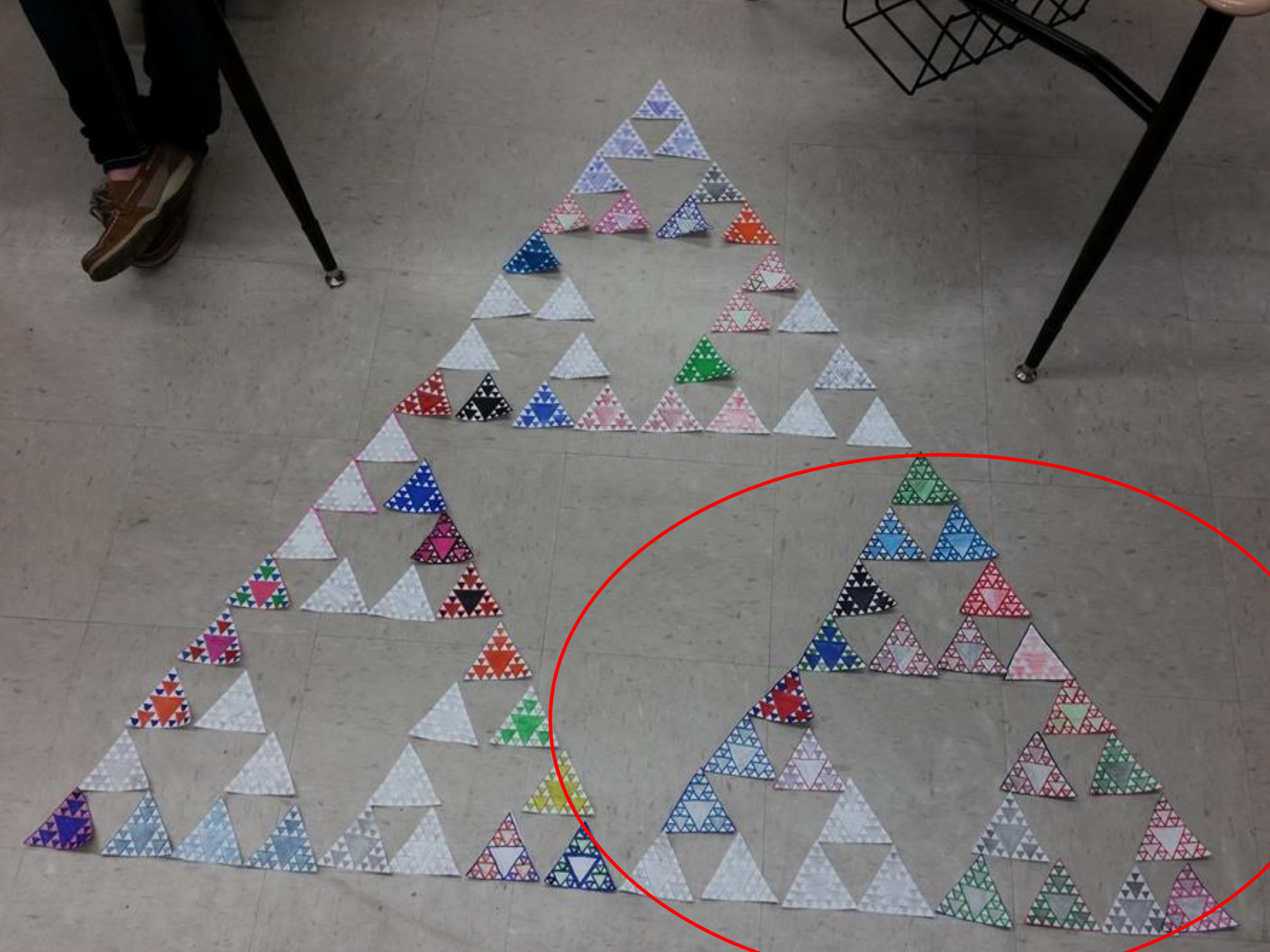
1,000 iterations?

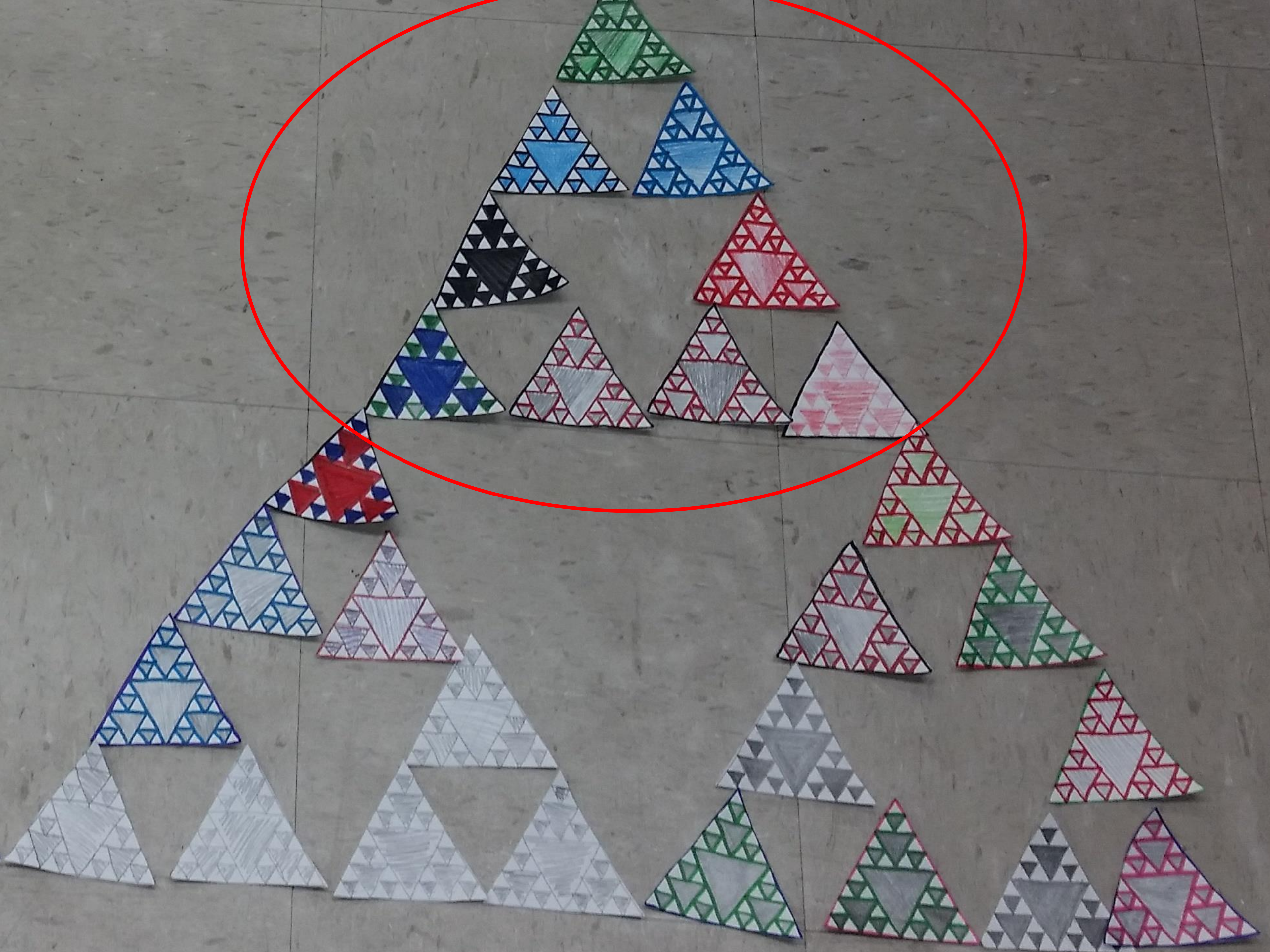
Why?

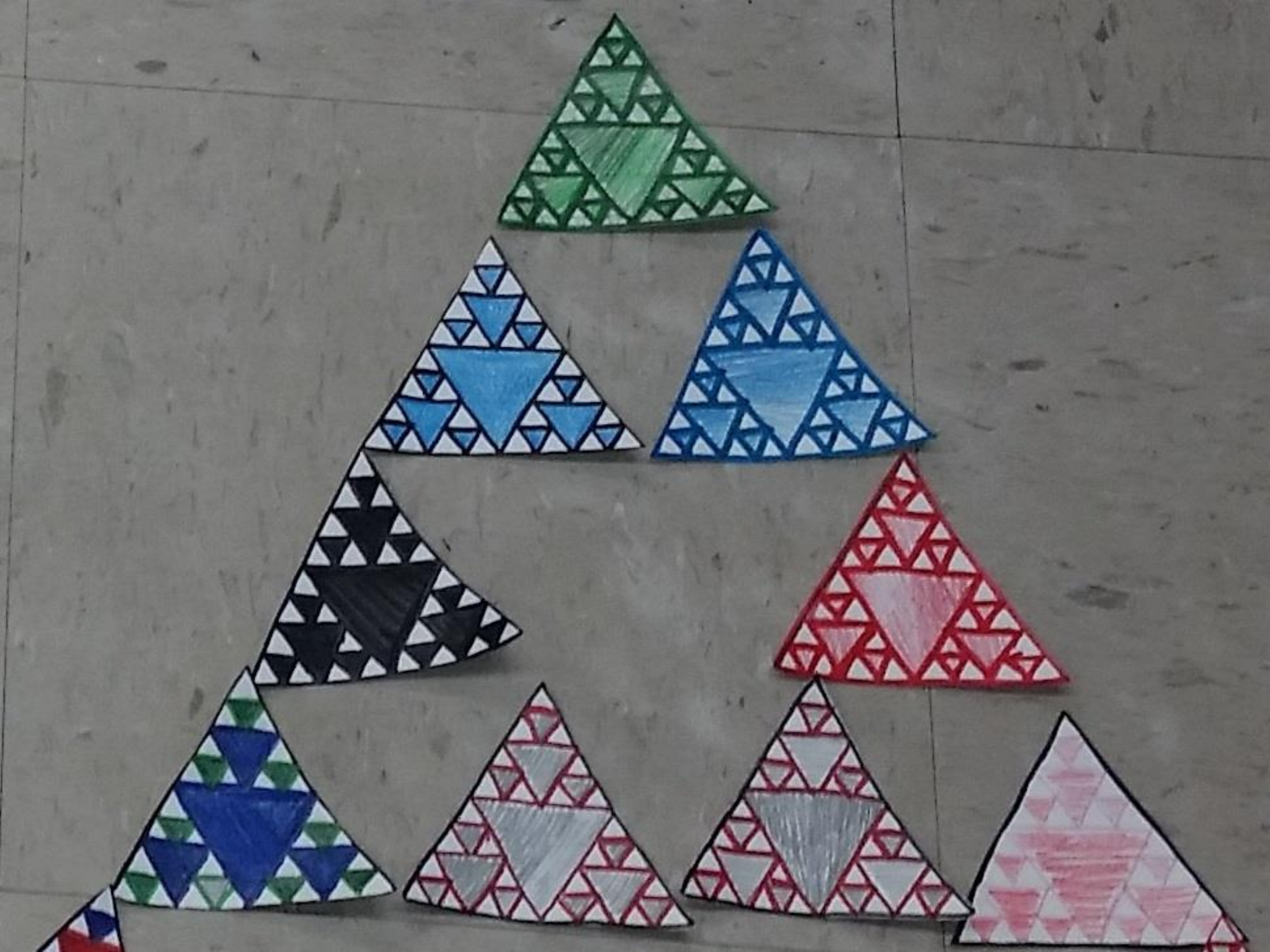
Let’s try it!

The Chaos Game on the TI-Nspire

From *www.johnhanna.us/TI-nspire.htm*

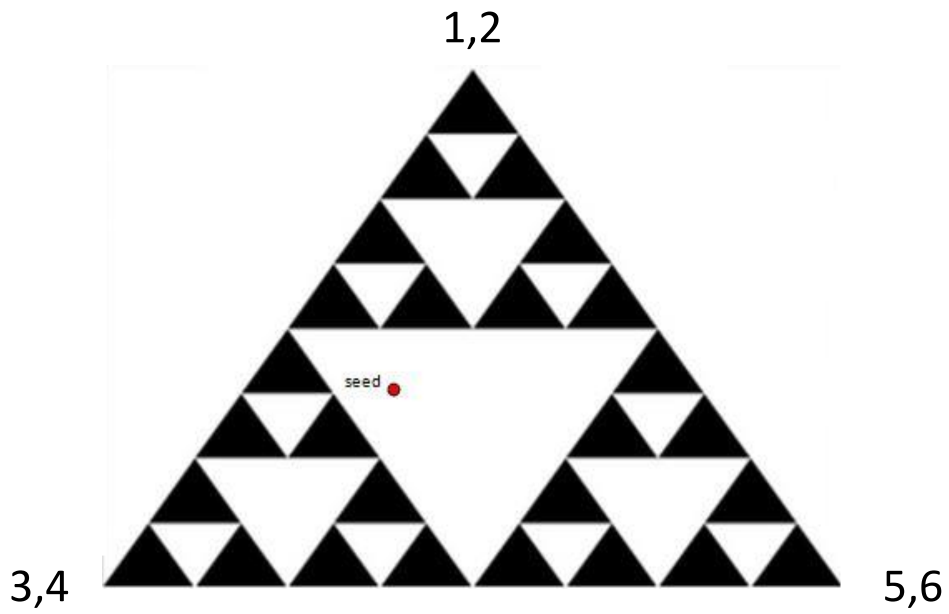




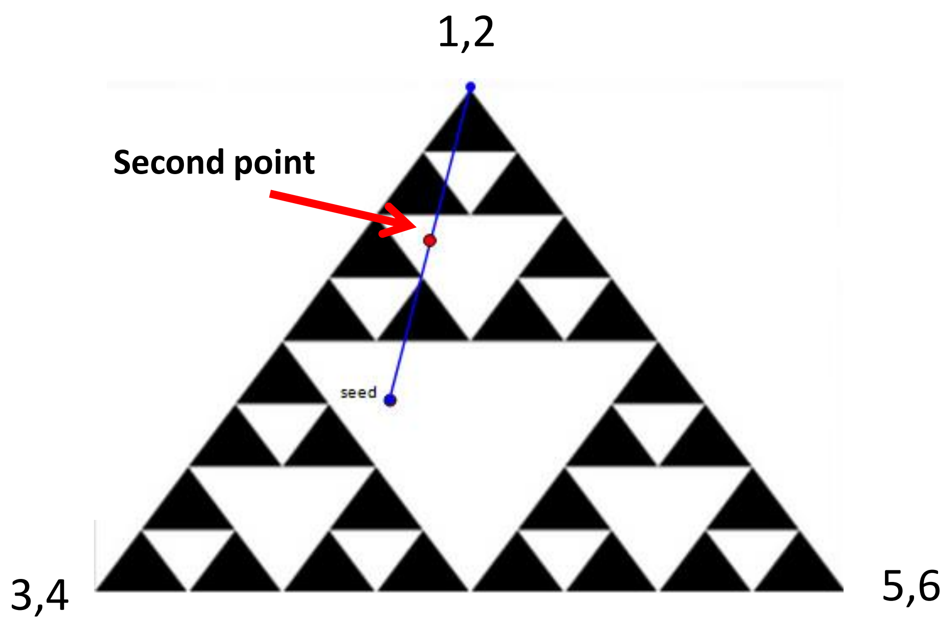


Why does the Chaos Game result in the Sierpinski triangle?

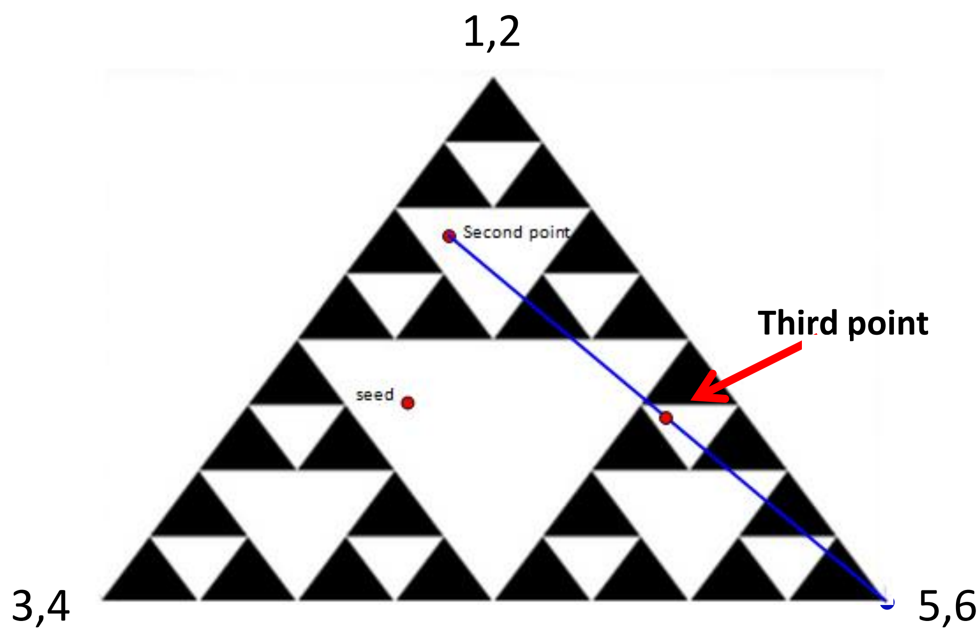
- Choose a seed point within the largest triangle removed from the Sierpinski triangle.



- Randomly choose a vertex, then plot the second point at the midpoint between the seed and chosen vertex.
- Where is the second point located? Will this always be the case?



- Again, randomly choose a vertex, then plot the third point at the midpoint between the second point and chosen vertex.
- Where is the third point located? Will this always be the case?



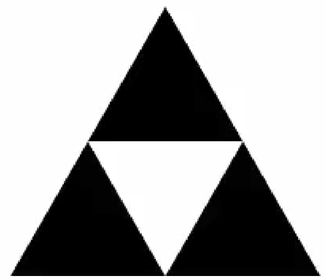
Why does the Chaos Game result in the Sierpinski triangle?

We see that the "orbit" or path of the point moves to **successively smaller triangles.** After a few iterations, the point enters a small triangle that is virtually impossible to see.

Thus, if we discard the first 8-10 points (iterations) of the Chaos Game, the points that are recorded will not be located in the triangular white "holes" of the Sierpinski triangle that are visible to the eye.

Why does the Chaos Game result in the Sierpinski triangle?

Note: The orbit will actually never reach the Sierpinski triangle (the black areas of the triangle that are left when the successively smaller white triangles are removed). Thus, the Sierpinski triangle is sometimes called a "strange attractor" because the orbit gets closer and closer to the Sierpinski triangle without ever reaching it.



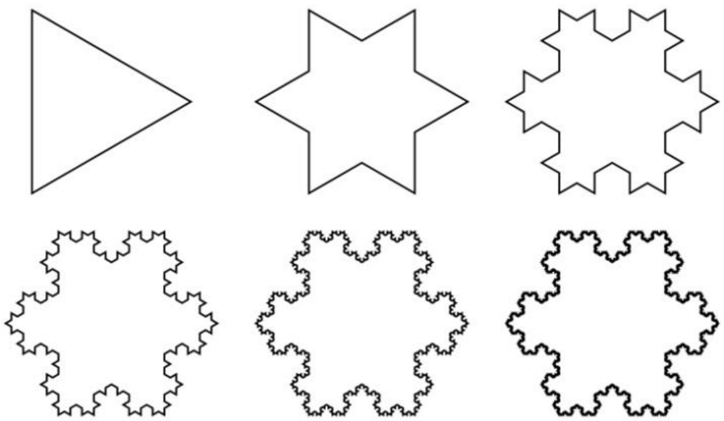
Other (Free) Technology Options

As we saw from the TI-Nspire activity, it takes quite a few iterations for a pattern to emerge. Here are some other options to run a simulation using technology.

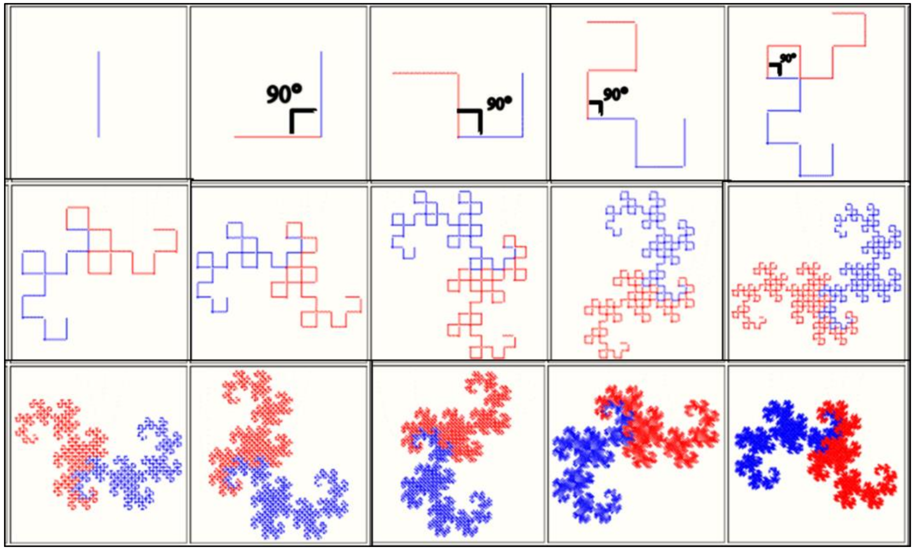
- Geogebra
- Python
- Shodor Interactivate
- Chaotica (iOS app)

Fractals that lend themselves to study in a secondary mathematics classroom:

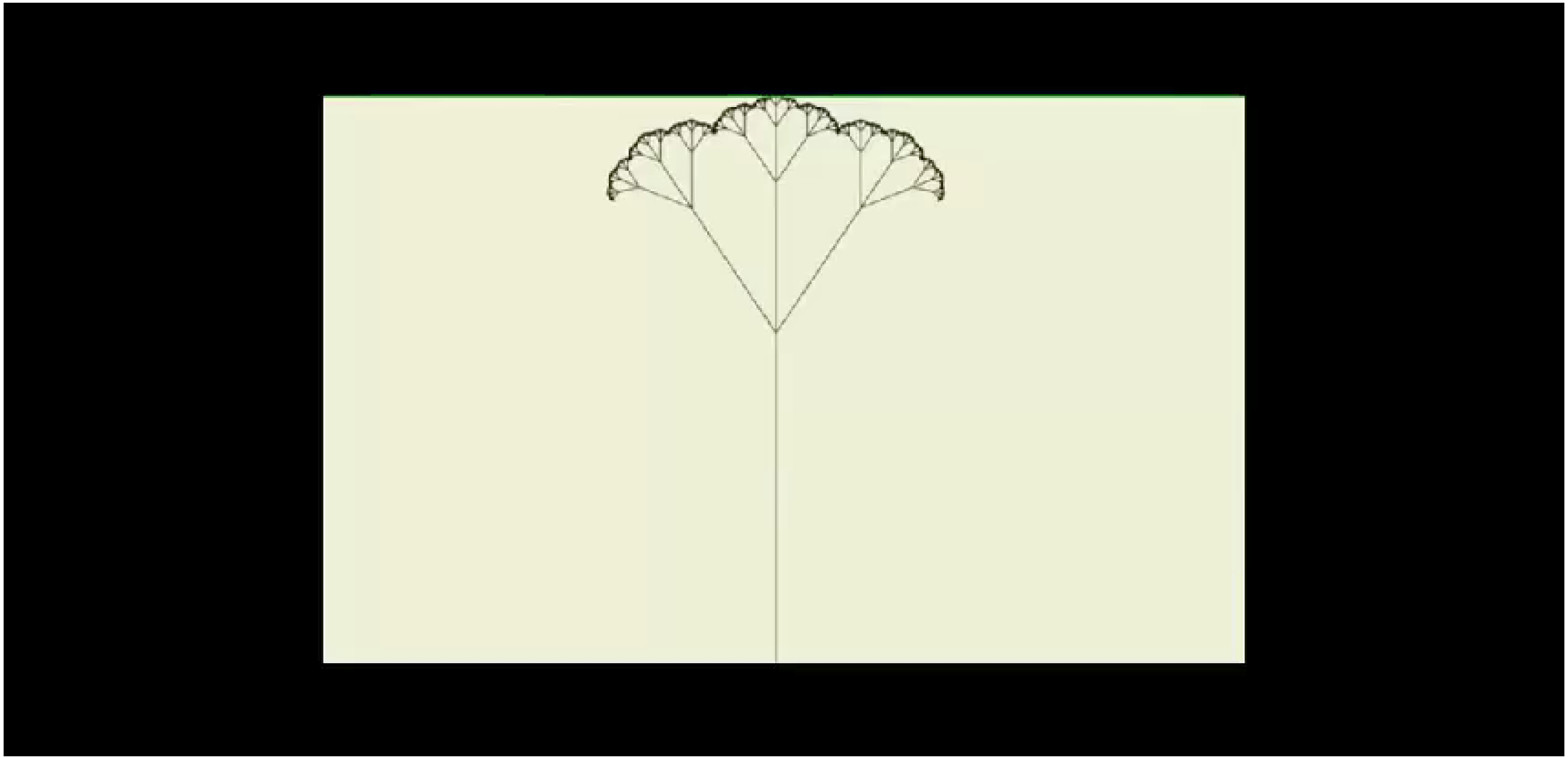
Koch snowflake



Jurassic Park fractal



Finding the Sierpinski triangle in surprising places!
(iOS app: Geom-e-Tree)



Rice University School Mathematics Project
Houston, Texas
Website: www.rusmp.rice.edu

Alice Fisher
afisher@rice.edu
Director of Technology Applications & Integration

Susan Troutman
troutman@rice.edu
Director of Secondary Programs

Session # 106