SIDE-SIDE-ANGLE CONDITIONS FOR CONGRUENT TRIANGLES: NEW ANSWERS FOR OLD PROBLEMS

1. COMMON CONDITIONS FOR CONGRUENT TRIANGLES

The topic about congruent triangles is a traditional part of Geometry courses. To decide whether two triangles are congruent or not, it is necessary to compare six elements – three sides and three angles in each triangle under consideration. But it turns out it is enough to compare only three elements – different combinations of sides and angles. Regular high school Geometry courses as usual comprise four such congruency statements – SAS, ASA, AAS, SSS, and HL. It is well known that all these statements are theorems. Their proofs are based on the fundamental geometric postulates. Nevertheless very often, in the high school textbooks, statements SAS, ASA, and SSS are considered as postulates [1], [2]. Among such congruency statements, the Side-Side-Angle (SSA) conditions are also considered in the courses of Geometry. Such statement describes the case when in two triangles there are two congruent sides and congruent non included angles (which are not between congruent sides). This question is a matter of many discussions and exercises [3], [4]. It is easy to draw pictures when such initial SSA conditions lead to two different triangles which are obviously not congruent. The current paper investigates the question when triangles with SSA conditions can be congruent. It possible to find such conditions and prove the triangles which satisfy these conditions are congruent.

2. CONGRUENCY DEFINITIONS AND PREREQUISITES

Before to discuss and proof the theorem, it is necessary to discuss some prerequisites which help to prove the theorem under consideration. Such prerequisites are:

a) The definition of congruent triangles
b) Types of proofs
c) Theorems or properties proved earlier.

Consider these aspects one by one.

a) In Geometry, definitions are an important part of the entire course because it is definitions that make a basis for the discovering of new properties of geometric figures. Mathematical definitions could differ significantly from the intuitive understanding which people use in their everyday life. Such intuitive definition of congruency could be as follows:

Congruent triangles have corresponding sides and corresponding angles congruent [1, p.225].

Although such definition is clear from our everyday experience, it is ineffective for proofs. Such definition does not give the hint how to begin the proof and how to determine congruent sides or angles in triangles except the direct measurements. But the direct measurements cannot be basis for the theoretical proof of the theorem because the number of all possible figures (triangles) is infinite.

Effective proofs of theorems about congruent triangles is based on the matching triangles on each other and describing they coincide in al points. Namely this way of proof for the SAS statement was used by Euclid [5, p.247]. Let us follow the same method for our purposes. For this reason, the congruency definition should be accepted in the following way:

Definition: *Triangles are congruent if they coincide in all points after matching them on each other.*

Such definition, which can be called as a *superposition principle*, is effective because it makes the proof of the theorem possible. The matter of such proof is to describe why overlapping triangles have to coincide. Form such definition it follows that once triangles coincide (are congruent) then they have the same sides and angles. It turns out that the previous (ineffective) definition could be considered as the consequence or explanation of the definition based on the superposition principle.

b) There are different types of proofs, namely a direct proof and indirect proof indirect proof which is often is called a proof by a contradiction. The direct proof is a way to show that the basic statement is true by a combination of axioms and postulates or previously known facts and theorems. The indirect proof is based on the assumption that if some accepted statement leads to the wrong conclusion, then such statement is wrong and the negation of such statement has to be true. Such approach implies that either the statement or its negation has to be true. The proof below belongs to the indirect type of proof.
c) For the proof of the SSA theorem below we will use the fact that in the isosceles triangle, the base angles are both acute and congruent. Such property can be easily proved, and we will consider it as a given fact.

Now it is possible to discuss the SSA theorem itself.

3. SIDE-SIDE-ANGLE THEOREM

Consider different combinations of triangles which satisfy SSA conditions when these triangles have two congruent sides and one non included congruent angle (which is not between these congruent sides). It is possible to recognize 6 such combinations which are represented below.

1) In triangles ΔABC and ΔA’B’C’ there are two congruent sides AC and A’C’ and congruent sides BC and B’C’. These triangles have congruent acute non included angles <A and <A’. The included angle <C is obtuse and the included angle <C’ is acute. Third corresponding angle <B is acute and the angle <B’ is obtuse. This case is represented in the Fig. 1 below.

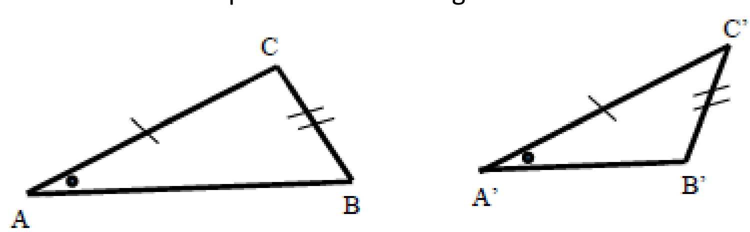
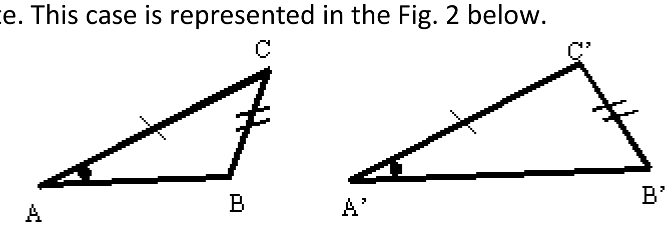


Fig. 1. Triangles that satisfy SSA conditions.

2) In triangles ΔABC and ΔA’B’C’ there are two congruent sides AC and A’C’ and congruent sides BC and B’C’. These triangles have congruent acute non included angles <A and <A’. The included angle <C is acute and the included angle <C’ is obtuse. Third corresponding angle <B is obtuse and the angle <B’ is acute. This case is represented in the Fig. 2 below.



3) In triangles ΔABC and ΔA’B’C’ there are two congruent sides AC and A’C’ and congruent sides BC and B’C’. These triangles have congruent acute non included angles <A and <A’. The included angle <C is obtuse and the included angle <C’ is obtuse. Third corresponding angle <B is acute and the angle <B’ is acute. This case is represented in the Fig. 3 below.

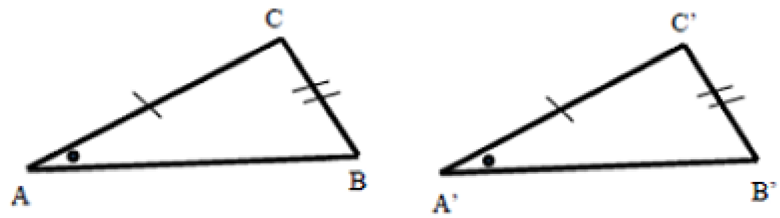


Fig.3. Triangles that satisfy SSA conditions.

4) In triangles ΔABC and ΔA’B’C’ there are two congruent sides AC and A’C’ and congruent sides BC and B’C’. These triangles have congruent acute non included angles <A and <A’. The included angle <C is acute and the included angle <C’ is acute. Third corresponding angle <B is acute and the angle <B’ is acute. This case is represented in the Fig. 4 below.

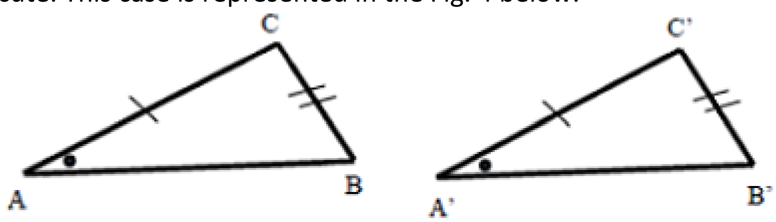


Fig.4. Triangles that satisfy SSA conditions.

5) In triangles ΔABC and ΔA’B’C’ there are two congruent sides AC and A’C’ and congruent sides BC and B’C’. These triangles have congruent acute non included angles <A and <A’. The included angle <C is acute and the included angle <C’ is acute. Third corresponding angle <B is obtuse and the angle <B’ is obtuse. This case is represented in the Fig. 5 below.

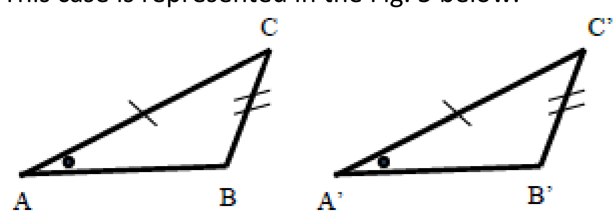


Fig.5. Triangles that satisfy SSA conditions.

6) In triangles ΔABC and ΔA’B’C’ there are two congruent sides AC and A’C’ and congruent sides BC and B’C’. These triangles have congruent obtuse non included angles <A and <A’. The included angle <C is acute and the included angle <C’ is acute. Third corresponding angle <B is acute and the angle <B’ is acute. This case is represented in the Fig. 6 below.

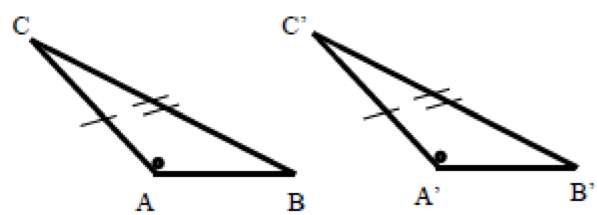


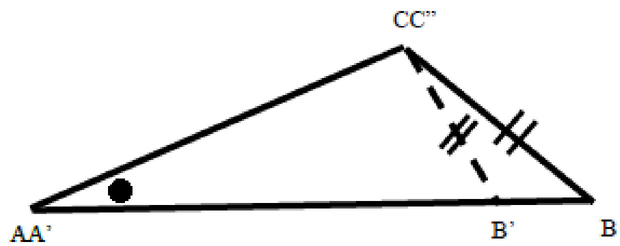
Fig.6. Triangles that satisfy SSA conditions.

Examples on Fig. 1-6 describe all possible case for triangles which satisfy the SSA conditions. In cases 1 and 2 triangles are obviously non congruent because they have different angles. It is clear that in cases 1 and 2 above triangles are not congruent despite they have two congruent sides and

angles. Other corresponding angles are of different types – if in one triangle the third angle is acute, then in another triangle the corresponding angle is obtuse and vise versa.
In the cases 3-6 triangles look as if they are congruent. In cases 3-6 the second and third angles are of the same type – if in one triangle the angle is acute (obtuse), then in another triangle the corresponding angle is also acute (obtuse).
Is it really true? The answer for such question gives the following

Theorem 1 (SSA). If *two sides and one non included angle in one triangle are congruent to two corresponding sides and the non included angle in another triangle* ***and*** *other corresponding angles are both acute or both obtuse, then such triangles are congruent.*

Proof. Consider for example the case 3. Let us move the triangle ΔA'B'C' on the triangle ΔABC and match sides AC and A'C' (see the Fig.7 below).



1. The side A'C' will coincide with the side AC because they are congruent.
2. The side A'B' will go along the side AB because the angle <C'A'B' and the angle <CAB are congruent.
3. Pretend as if sides CB and C'B' do not coincide and, for example, the side C'B' is inside the triangle Δ(AA')(CC')B. Then the triangle ΔB'(CC')B will be an isosceles triangle because CB and C'B' are congruent. Once angle (CC')B'(AA') is acute then the angle <(CC')B'B must be obtuse. But this is impossible because in the isosceles triangles the base angles are congruent and acute. We have the contradiction. So, the assumption that side CB and C'B' do not coincide is wrong. These sides must coincide. So, our triangles coincide in all sides. In other words, these triangles are congruent.

End of theorem.

All other cases 4, 5, and 6 can be proved the same way. Such proves can be also found in [6] and [7]. It is possible to represent all triangles that satisfy SSA conditions in the Table 1 below.

#	**Congruent angle**	**Included angle**	**Third angle**	**Congruent triangles**
1	acute	First triangle - obtuse Second triangle- acute	First triangle - acute Second triangle- obtuse	no
2	acute	First triangle - acute Second triangle- obtuse	First triangle - obtuse Second triangle- acute	no
3	acute	First triangle - obtuse Second triangle- obtuse	First triangle - acute Second triangle- acute	yes
4	acute	First triangle - acute Second triangle- acute	First triangle - obtuse Second triangle- obtuse	yes
5	acute	First triangle - acute Second triangle- acute	First triangle - acute Second triangle- acute	yes
6	obtuse	First triangle - acute Second triangle- acute	First triangle - acute Second triangle- acute	yes

Table 1. Possible combinations of triangles which satisfy the SSA conditions.

In the traditional statements about congruent triangles (SAS, ASA, and SSS) there are three elements to compare. In the SSA statement, for the initial three conditions about the congruency of two sides and one angle it is necessary to know the type of other two angles (obtuse or acute) but their measure whether these measures are equal. So, it is not necessary to compare all six parts of triangles – three sides and three angles.

4. Application of Side-Side-Angle Theorem

The proved Side-Side-Angle theorem urges to reevaluate answers for many problems considered to be solved. Some examples below demonstrate how SSA statement enables to find new answers for problems about congruent triangles.

(1) First example is the problem 9 from [1, p.243]. Question: decide whether enough information is given to prove that the triangles are congruent. If there is enough information, state the congruence postulate or theorem you would use. See the Fig. 8 below.

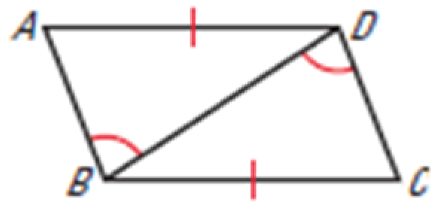


Fig. 8. Congruent triangles due to SSA, case 5.

It is clear that none of traditional statements SAS, ASA, AAS, or SSS can be applied here. So, the answer is supposed to be “There is no enough information to decide triangles ΔBDA and ΔDBC are congruent”. The question about the congruency remains opened.

Nevertheless the SSA theorem cab be applied her. It is exactly the case 5 from the Table 1. The triangles ΔBDA and ΔDBC have two congruent sides and non included congruent angles. Other corresponding angles are acute. So, such triangles are certainly congruent. The proof of the congruency could be similar what was done before.

Let us mark the triangle ΔDBC as ΔD’B’C as represented in the Fig. 9.

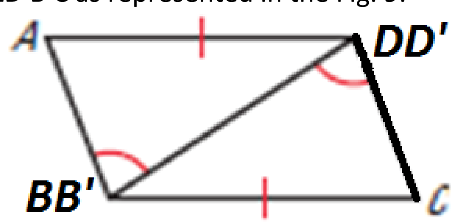


Fig. 9. Renaming the triangle ΔDBC as ΔD’B’C.

Triangles ΔBDA and ΔD’B’C can be matched after the combination of a reflection, a translation, and a rotation. The segment BD will coincide with the segment B’D’ because they have the same length. The segment DC will go along the segment BA because they have the same angles <ABD and <CDB. Suppose as if sides AD and CB’ do not coincide as it represented in the Fig. 10 below.

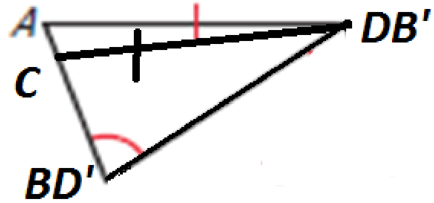


Fig. 10. Triangles after matching (not drawn in scale).

In such case it turns out that in the isosceles triangle ΔA(DB')C the angle <DAB is acute and the angle <AC(DB') is obtuse as a supplement to the acute angle <CB'D'. Such state is impossible because base angles in the isosceles triangle are congruent and acute. There is a contradiction. So, the assumption that sides AD and CB' do not coincide is wrong. These sides have to coincide. Once they coincide, then our triangles coincide in all point and consequently they are congruent.

The correct answer for the initial problem is the following – "There is enough information to conclude the given triangles on the Fig. 8 are congruent. The SSA theorem can be applied here".

(2) The next example is the problem 20 from [1, p.244] as shown in the Fig. 11. The question is decide whether enough information to p[rove the triangles are congruent. If there is enough information, state the congruence postulate or theorem you would use.

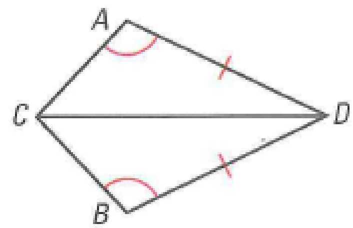


Fig. 11. Are such triangles congruent?

In such case again none of traditional statements (SAS, ASA, AAS, or SSS) cab be used. The question about the congruency of such triangles remains uncertain.

However, these triangles belong to the case 6 in the table 1. They have to be congruent. To make this prove, let us mark vertices of the triangle ΔDCB as ΔD'C'B as in the Fig. 12.

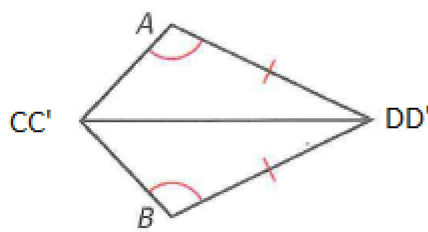


Fig. 11. Different triangles have different names for their vertices.

Now we can "cut out" the triangle ΔD'C'B and put it on the triangle ΔDCA along the side DA. Sides D'B and DA will coincide because they have the same length. The side B'C will go along the side AC because angles <A and <B are congruent. Imagine as if sides DC' and DC do not coincide as represented on the Fig. 13.

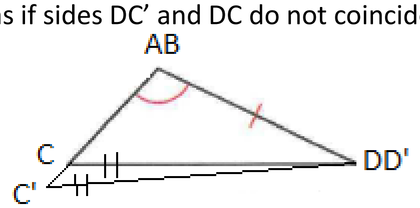


Fig. 13. Assumption that sides DC' and DC do not coincide.

In this case the isosceles triangle Δ(DD')CC' has no equal base angles. Namely, if angles <ACD and BC'D' are acute, then the angle C'CD' has to be obtuse. But this is impossible because it is a base angle in the isosceles triangle. So, the assumptions that sides DC' and DC do not coincides leads to the contradiction. It follows that sides DC' and DC do coincide. That means that triangles ΔDCA and ΔD'C'B coincide. In other words, such triangles on the Fig. 11 are congruent due to the SSA theorem.

Remark. The reflection of the triangle ΔDCB in the side CD will not enable to prove the statement. It is doubtful to prove that vertices A and B' will coincide. This case is represented in the Fig. 14 below. So, the correct proof should begin with the matching sides DA and DB rather than folding along CD.

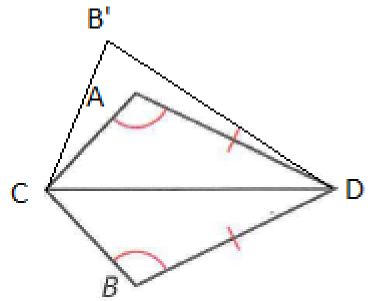


Fig. 14. Reflection in the side CD does not lead to the proof.

(3) Consider the problem 2 from [8, p.156]. The picture for this problem is represented in the Fig. 15. The question is whether triangles ΔRDH and ΔREG are congruent or not. It appears the answer is NO because neither SSS nor SAS theorems can be applied here. But from the case 5 in the Table 1 it follows such triangles are congruent and the answer is YES. These triangles have two congruent sides and non included vertical angles <DRH and <ERG. Other corresponding angles are acute. So, such triangles are congruent due to SSA theorem.

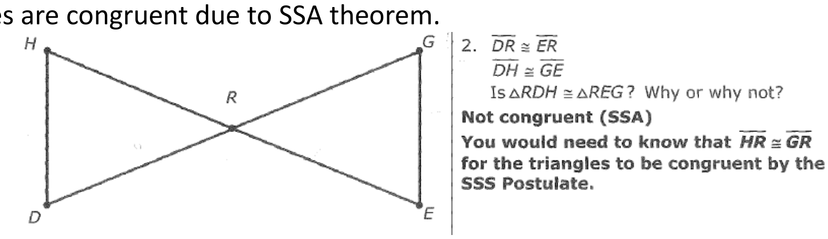


Fig.15. Example of congruent triangles.

(4) The fourth example is the problem #28 from [2, p.248]. The question is – which of the three triangles below can be proven congruent by SSS or SAS statement?

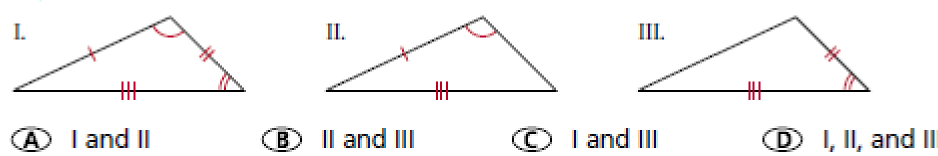


Fig. 16. Congruent triangles due to SSS or SAS statement.

It is clear the triangles I and III are congruent due to SAS statement. The correct answer is supposed to be **A**. But the triangles I and II correspond to the case 6 in the Table 1 above. It turns out they are also congruent due to SSA theorem. So, by transitivity, all these triangles are congruent.

5. CONCLUSIONS

The above examples show the effectiveness and the consistency of the Side-Side- Angle theorem. This theorem enables to find new answers for problems which are considered to be solved. It is possible to find more examples when the SSA theorem gives certain answers for problems considered to be uncertain or ambiguous. For this reason the SSA statement can be incused into the list of statements about congruent triangles in the Geometry curriculum.

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