Graphs are Everywhere!

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Computational and Applied Mathematics
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Summer Math Days
Rice University

June 2, 2015

Me and Mathematics

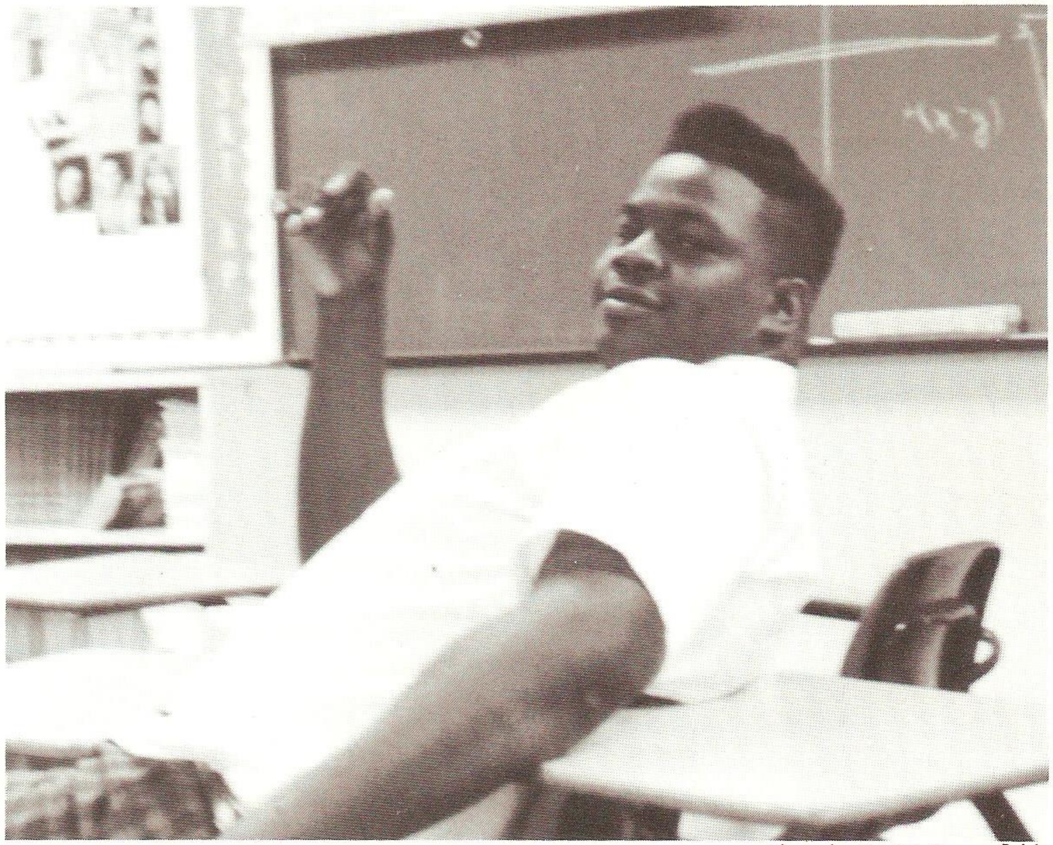


photo by Patricia Sappenfield

Senior Student Council members Harold Jackson and Alan Shipp discuss plans for a project after a Student Council meeting.

Senior Illya Hicks, a member of the offensive line and an Academic All-American, sits back and relaxes during calculus.

My Story

RICE
Unconventional Wisdom

TEXAS STATE UNIVERSITY
SAN MARCOS

ATM | TEXAS A&M UNIVERSITY

I Love Texas

MISSOURI
OKLAHOMA
ARKANSAS
NEW MEXICO
Canadian
Amarillo
Wichita Falls
Lubbock
Red
Texarkana
Denton
Dallas
Longview
Big Spring
Abilene
Ft. Worth
Tyler
Guadalupe Mts. Nat'l. Pk.
Midland
TEXAS
Waco
Sabine
El Paso
San Angelo
Killeen
Temple
Colorado
Rio Grande
AUSTIN
Beaumont
Houston
Del Rio
San Antonio
Galveston
Big Bend Nat'l. Pk.
LOUISIANA
Corpus Christi
Laredo
Padre I.
MEXICO
Gulf of Mexico
McAllen
Brownsville

I also still love football!

Outline

I. Basic Definitions

II. Different Graph Applications

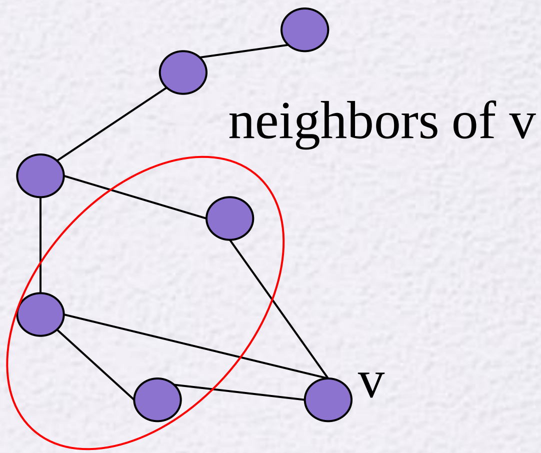
III. Dominating Sets, TSP, Clique & *k*-plexes

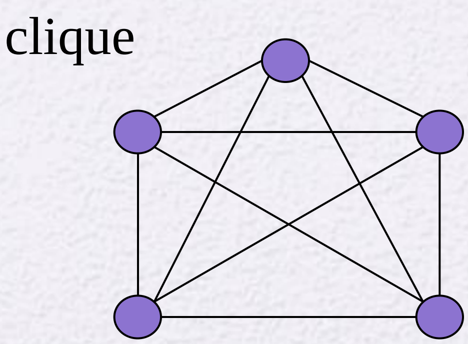
IV. Conclusions

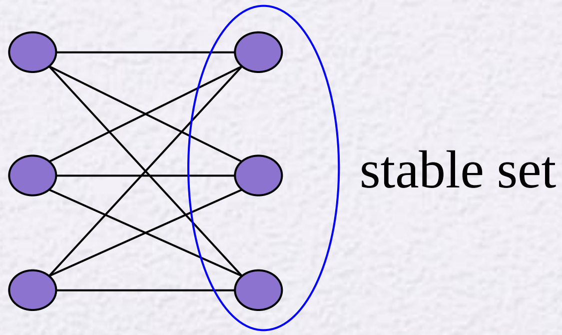
Graphs (Networks)

Graph G=(V, E)

- Vertex set V is finite
- Edges E = {uv : u,v ∈ V}
- Undirected (for this talk)
- u is a neighbor of w if uw ∈ E







I Can Tell You My Secret Now?

I see graphs everywhere!

Network (Graph) Applications

- vertices represent actors: people, places, companies
- edges represent ties or relationships
- Applications
  - Criminal network analysis
  - Data mining
  - Wireless Networks
  - Genes Therapy
  - Biological Neural Networks

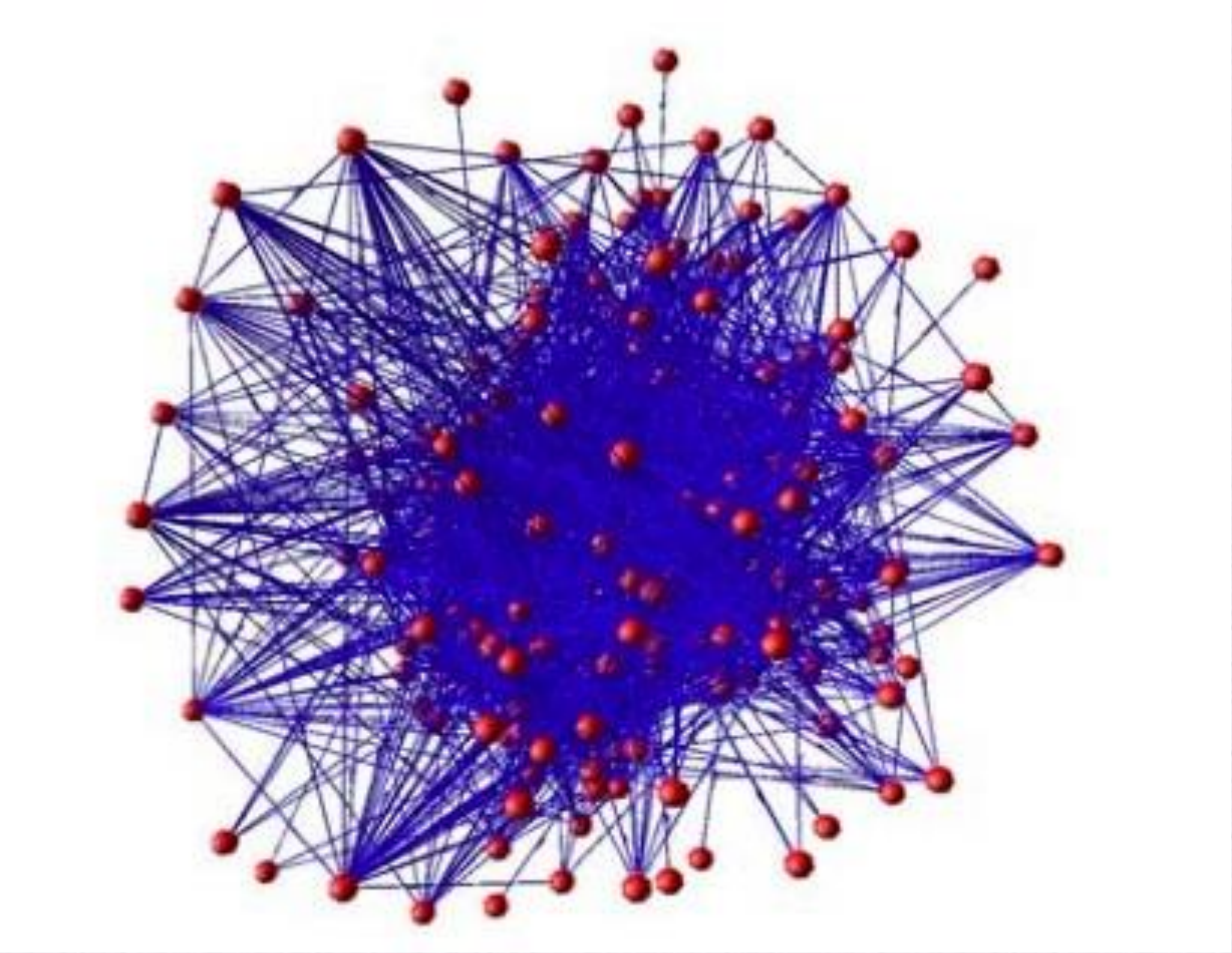
Van Gogh Graph

Provided by Don Johnson, Rice

Gene Co-expression Networks

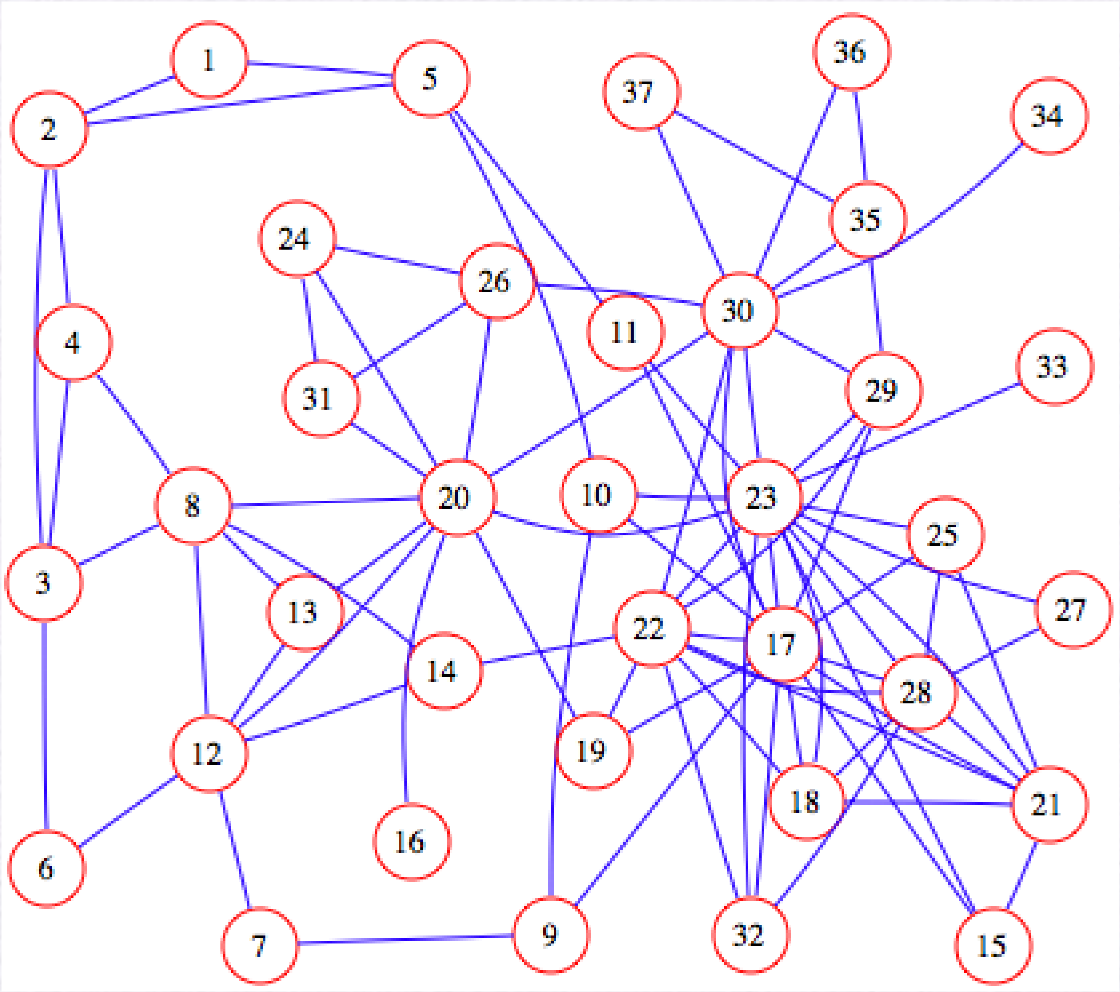
vertices represent genes
edges represent high correlation between genes
(Carlson et al. 2006)

Biological Neural Networks



vertices represent neurons
(Berry and Temman 2005)

Social Network Pop Quiz



9-11 Terrorist Network

1) Alshehri
2) Sugami
3) Al-Marabh
4) Hijazi
5) W. Alshehri
6) A. Alghamdi
7) M. Alshehri
8) S. Alghamdi
9) Ahmed
10) Al-Hisawi
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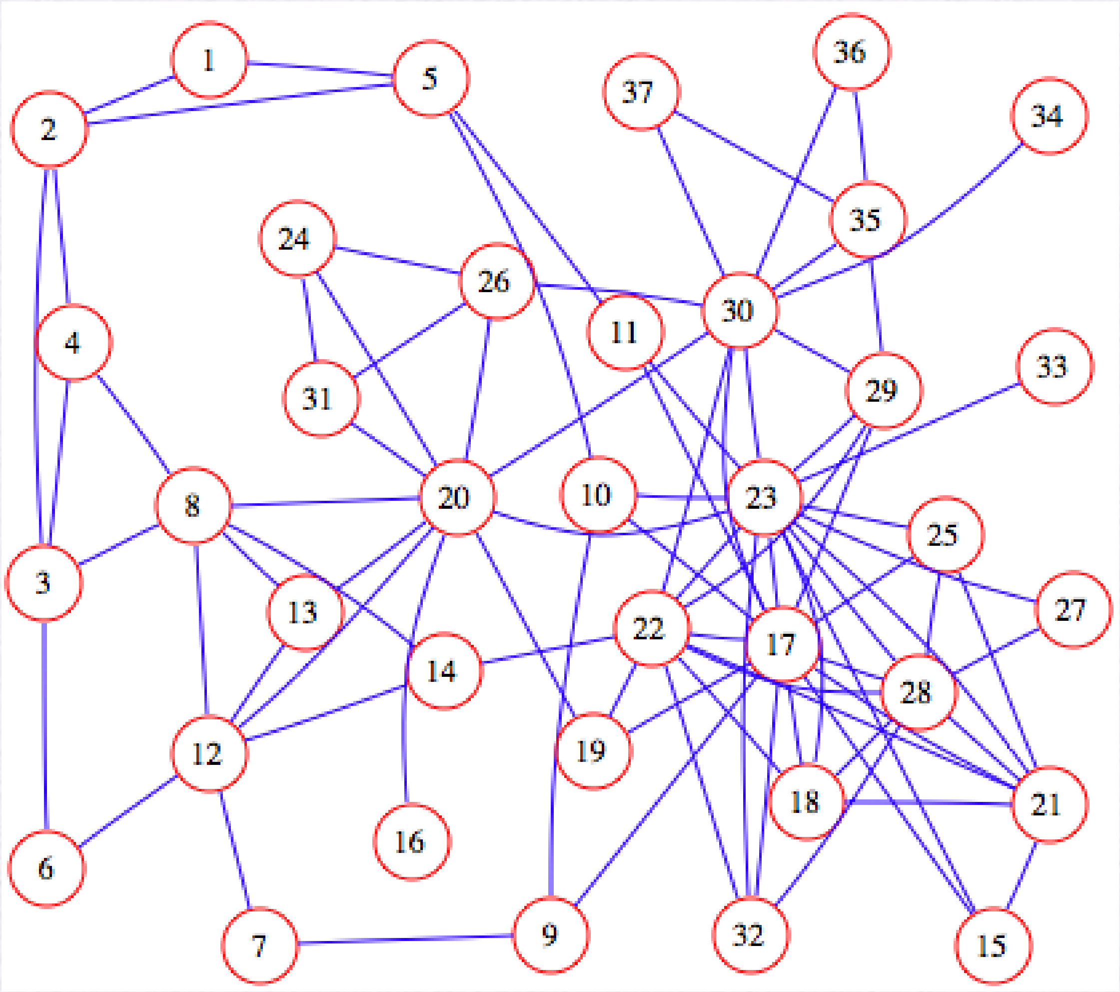
18) Essabar
19) S. Alhazmi



20) N. Alhazmi
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24) Shaikh
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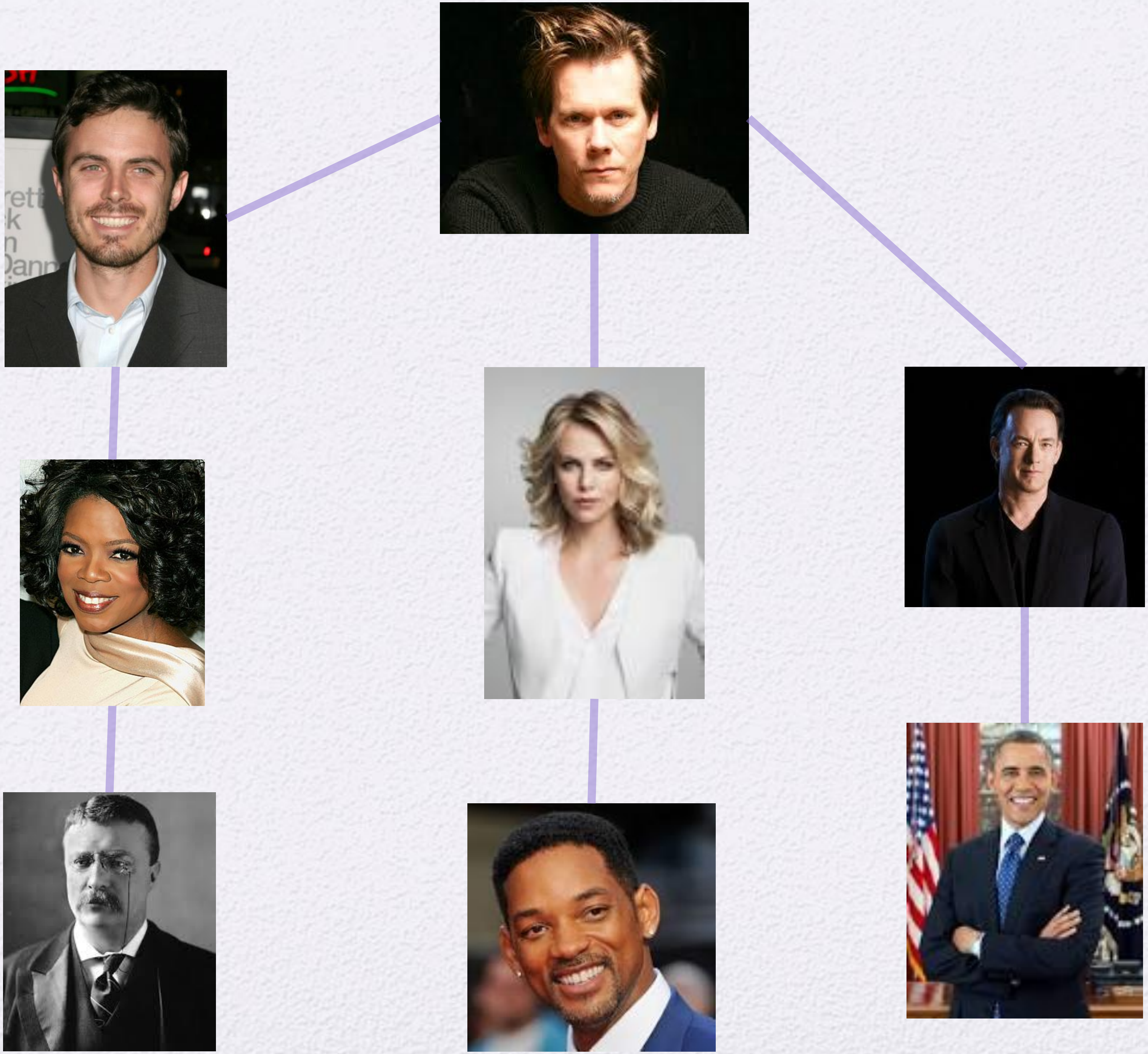
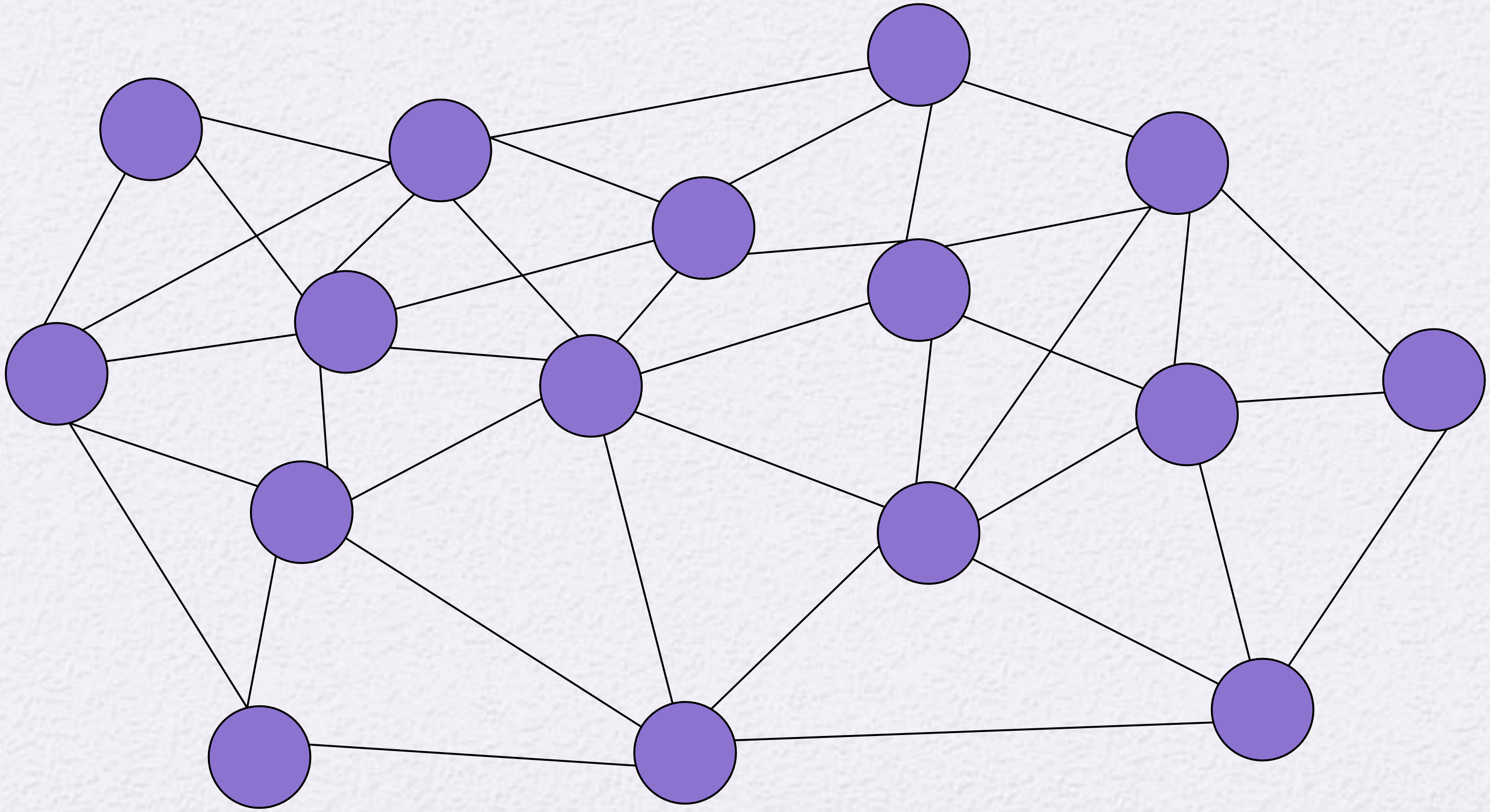
28) Al-Shibh
29) Raissi
30) Hanjour
31) Awadallah
32) Budiman
33) Al-ani
34) Moqed
35) Abdullah
36) Al Salmi
37) Alhazmi

Do You Like Bacon?



Dominating Set

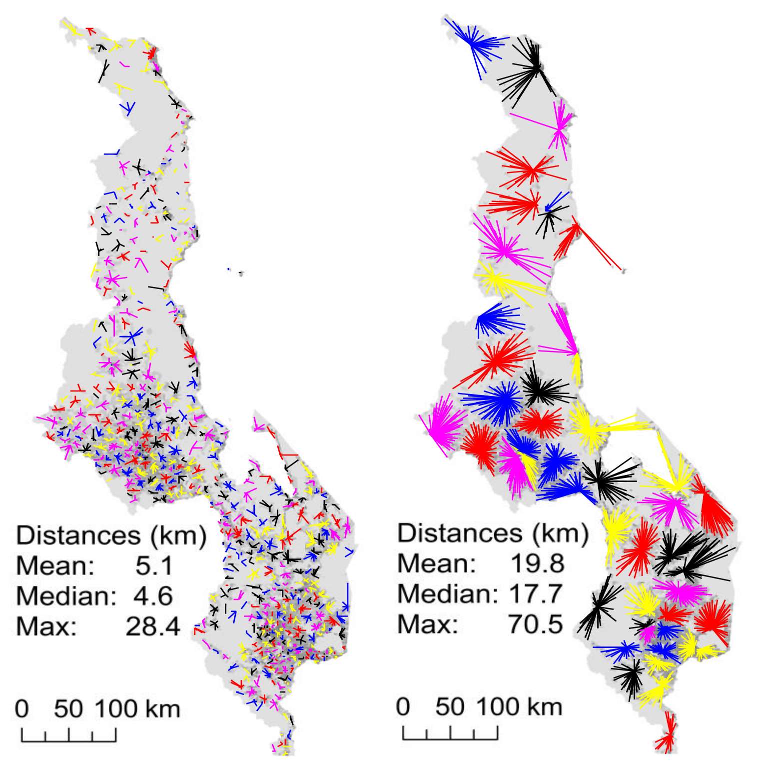
Dominating Set

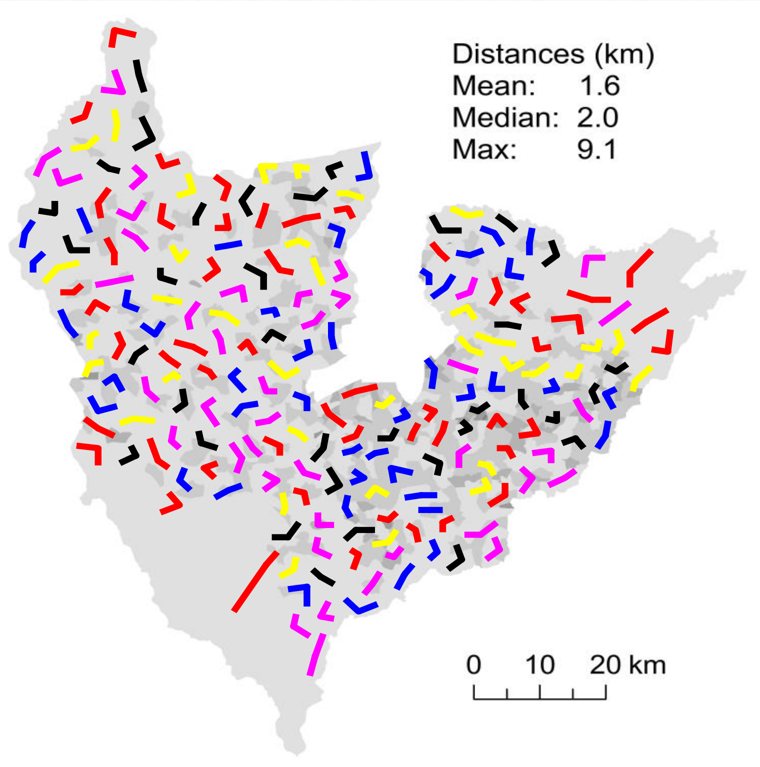


Minimum Dominating Set

- A dominating set D is a subset of vertices in a graph G such that every vertex of G is either a member of D or is adjacent to a member of D
- Applications
  - Sensor Networks
  - Marketing
  - Ad-hoc mobile networks (robots, cell phones)
  - Ship warehouse design

Health Logistics



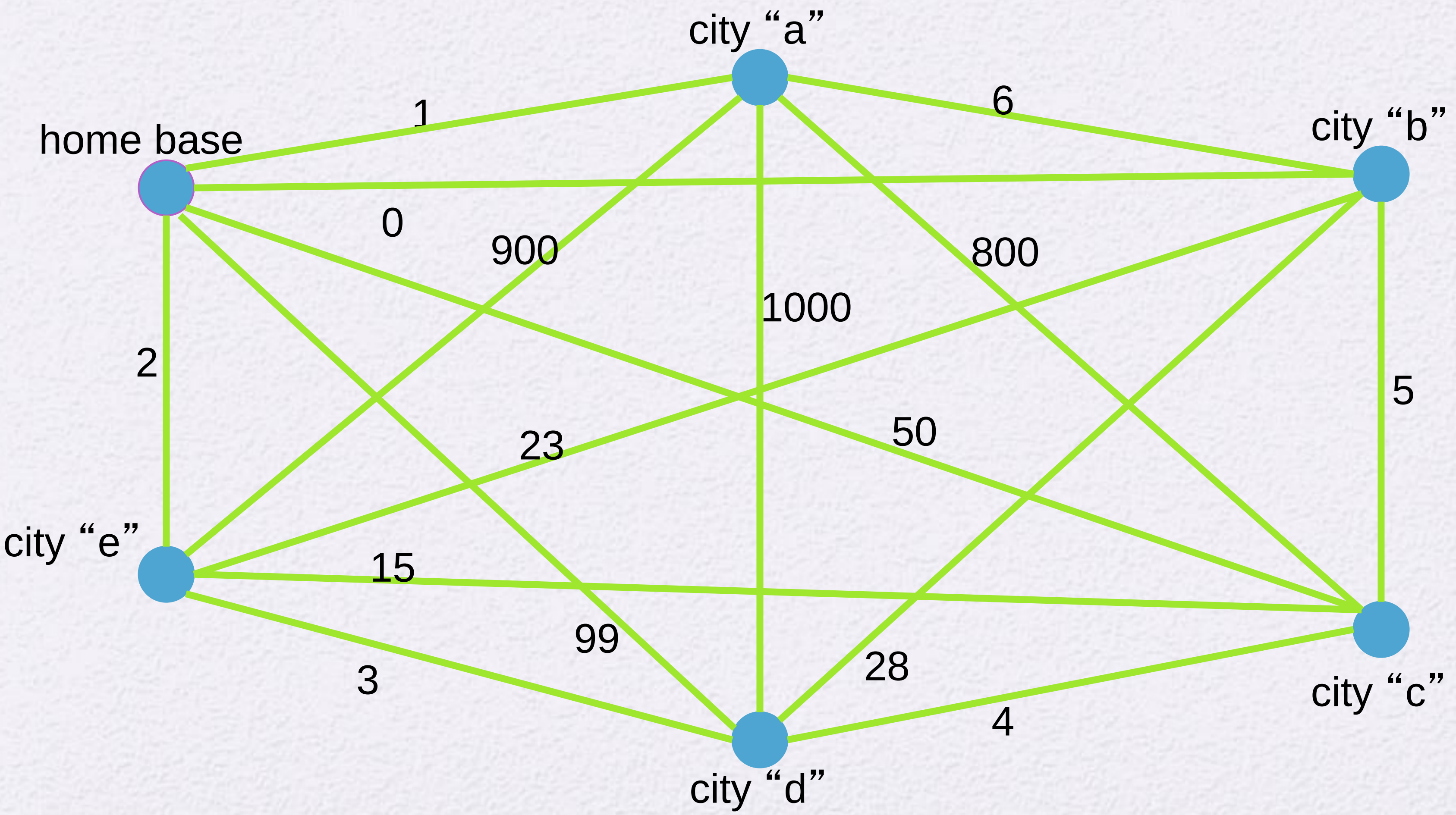


Amber Kunkel, Elizabeth Van Itallie, Duo Wu

Mission Impossible: Rogue Nation

- IMF instructions to Ethan Hunt:
  - Starting from home base, visit cities $\{c_2, c_3, \ldots, c_n\}$ to do covert operations and come back to home base.
  - You can not visit any city twice!
  - Since the agency is under budget cuts, you must complete your mission with lowest possible travel distance.

An Example


city "a"
home base
city "b"
1
6
0
900
800
1000
2
5
23
50
city "e"
15
99
28
3
4
city "c"
city "d"

Complexity of the Mission

- In general, there are $(n-1)!/2$ possible solutions.
- Suppose you could evaluate a possible solution in one nanosecond (10^{-9} seconds). If the number of cities were 23, then it would take you 178 centuries to look at all possible solutions.

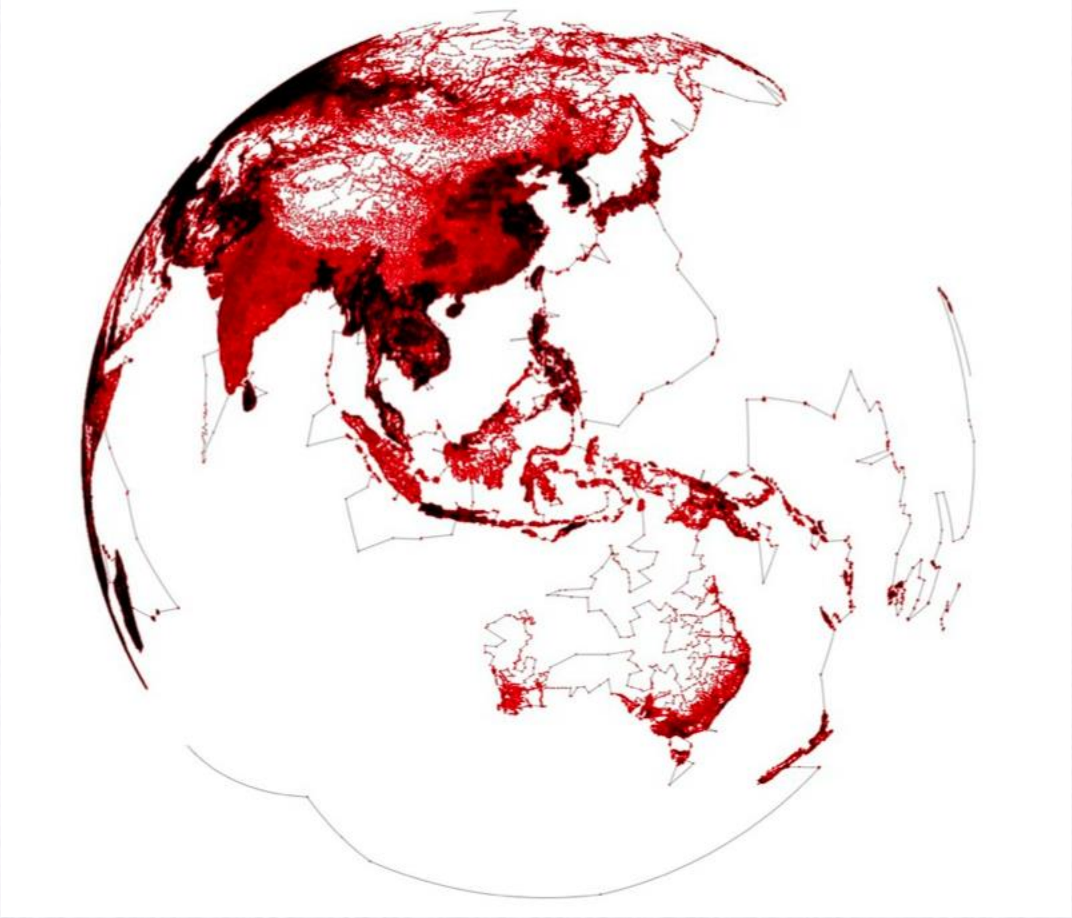
The Traveling Salesman Problem

Given a finite number of “cities” along with the cost of travel between each pair of them. Find the cheapest way to visit all the “cities” and return to your starting point.

Cities	Who?	Year
49	Dantizig, Fulkerson, and Johnson	1954
60	Held and Karp	1970
532	Padberg and Rinaldi	1987
2392	Padberg and Rinaldi	1988
7397	Applegate, Bixby, Chvatal, and Cook	1994
13,509	Applegate, Bixby, Chvatal, and Cook	1998

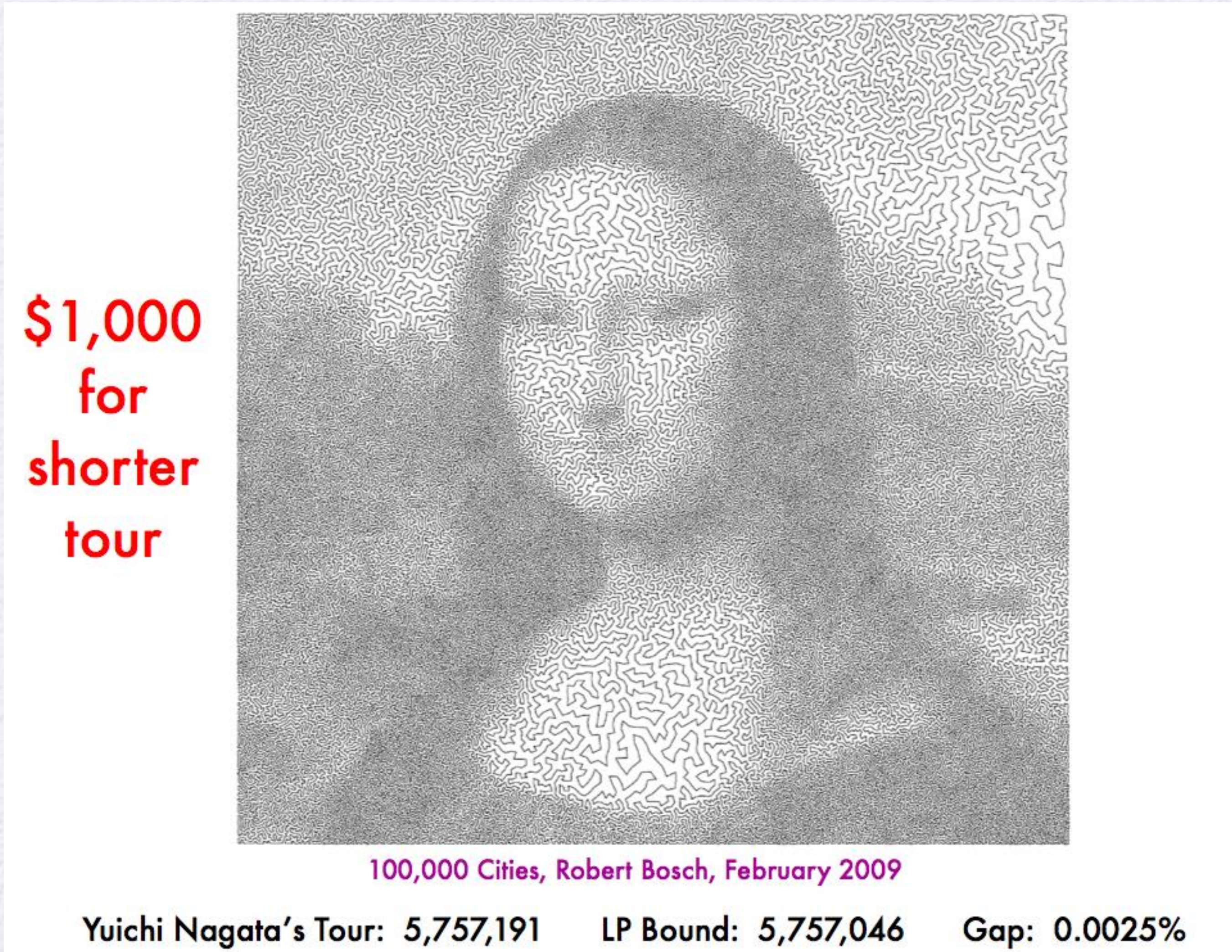
World TSP

World TSP: All 1,904,711 cities, towns, and villages. Created in 2001.



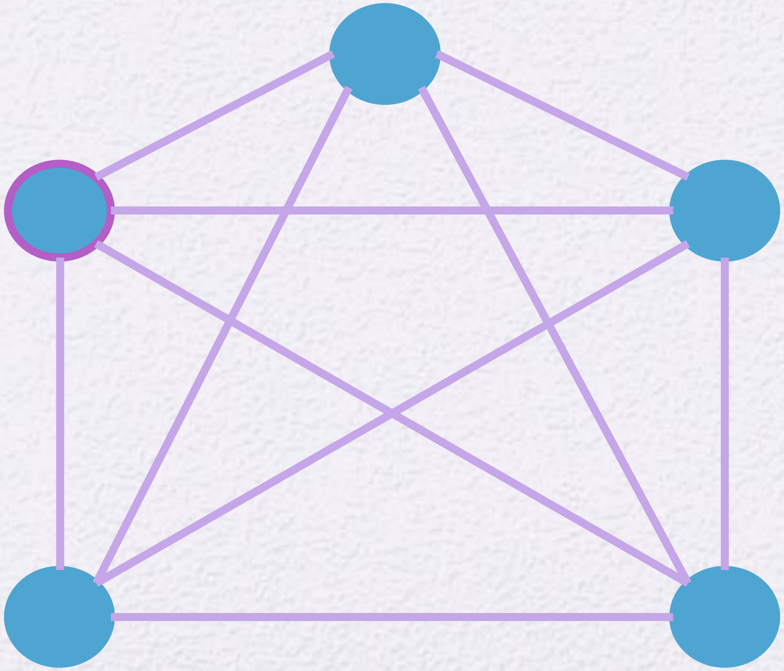
Keld Helsgaun's Tour: 7,515,790,354 LP Bound: 7,512,218,268 Gap: 0.0476%

Mona Lisa


$1,000
for
shorter
tour
100,000 Cities, Robert Bosch, February 2009
Yuichi Nagata's Tour: 5,757,191
LP Bound: 5,757,046
Gap: 0.0025%

Cliques

- A graph is a clique if every vertex is adjacent to the rest of vertices



Cliques

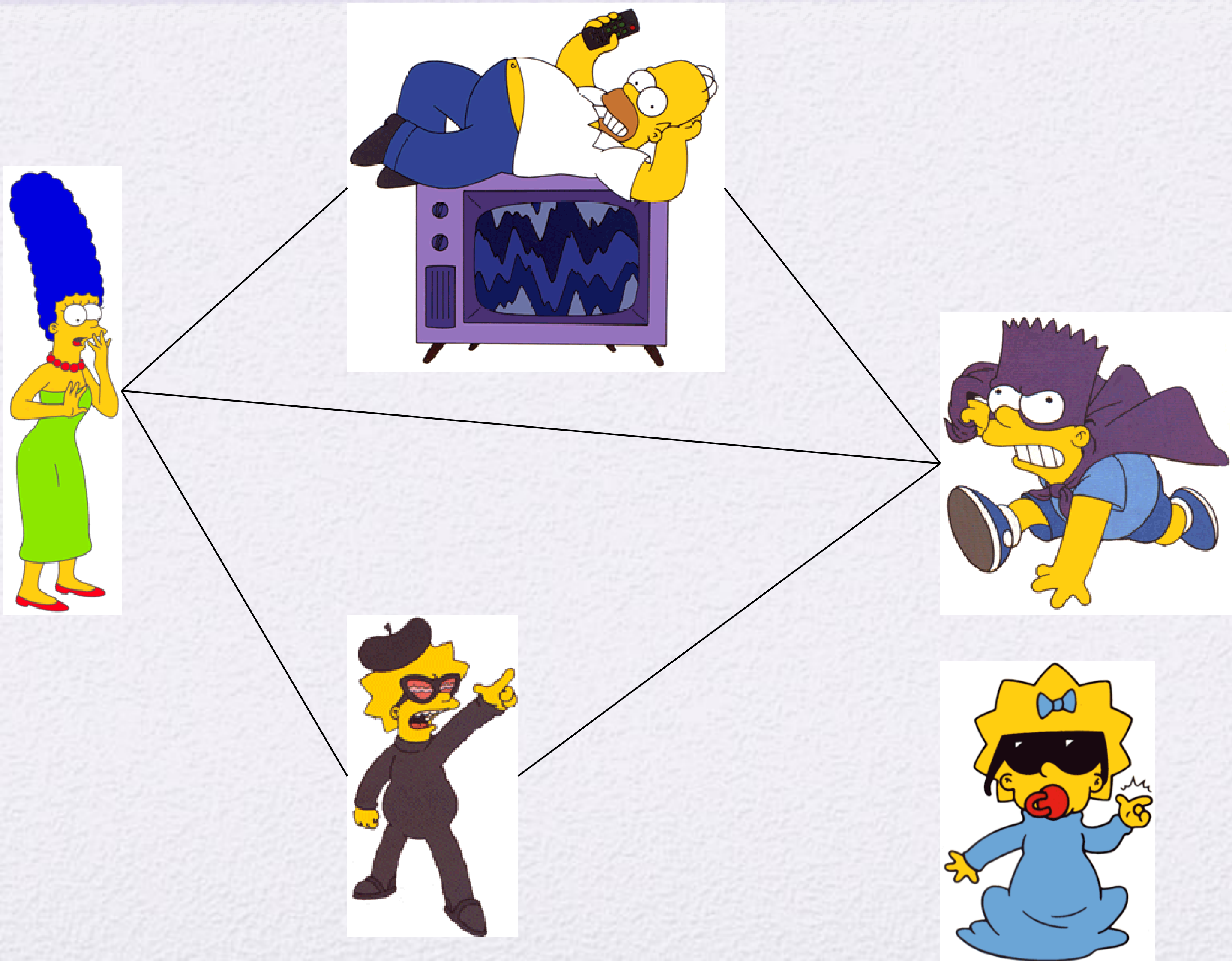
Maximum Clique

- A clique is a subset of nodes such that there is an edge between any two nodes in the set.
- two nodes can't be in a clique together if they are not adjacent
- Applications
  - Bioinformatics
  - Social networks
  - Online auctions

Homer Ignoring Lisa

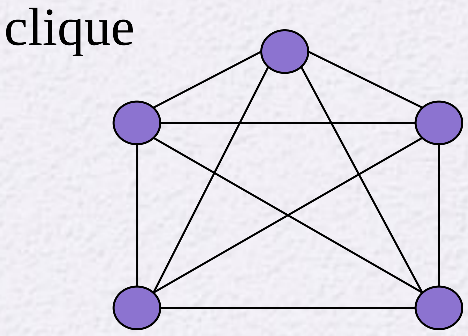
Homer ignoring Lisa (en espanol)

The Simpsons Social Network

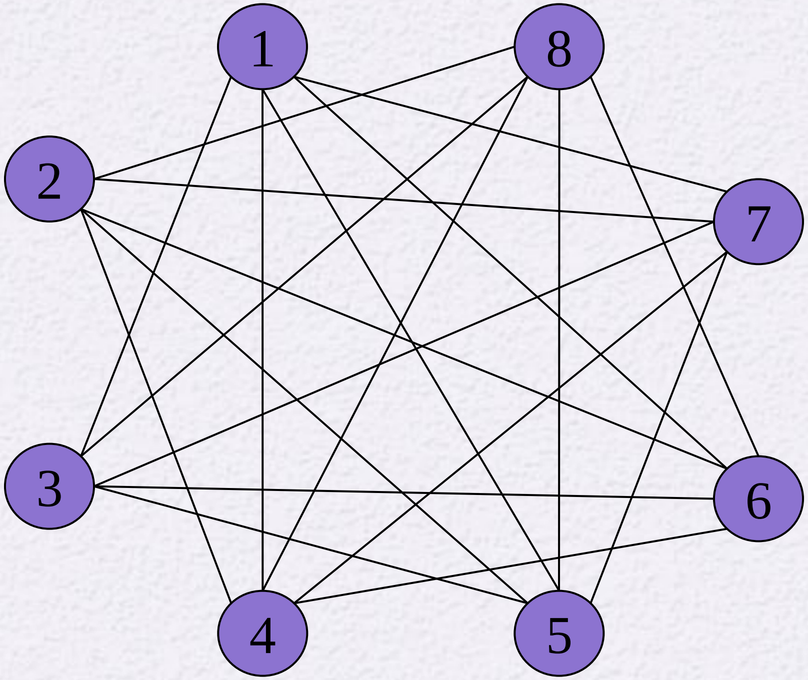


What is cohesiveness in terms of graphs?

- Debated by social scientists
- Three general properties
  - Familiarity (few strangers)
  - Members can easily reach each other (quick communication)
  - Robustness (not easily destroyed by removing members)



Is this graph cohesive?



Clique is too restrictive!

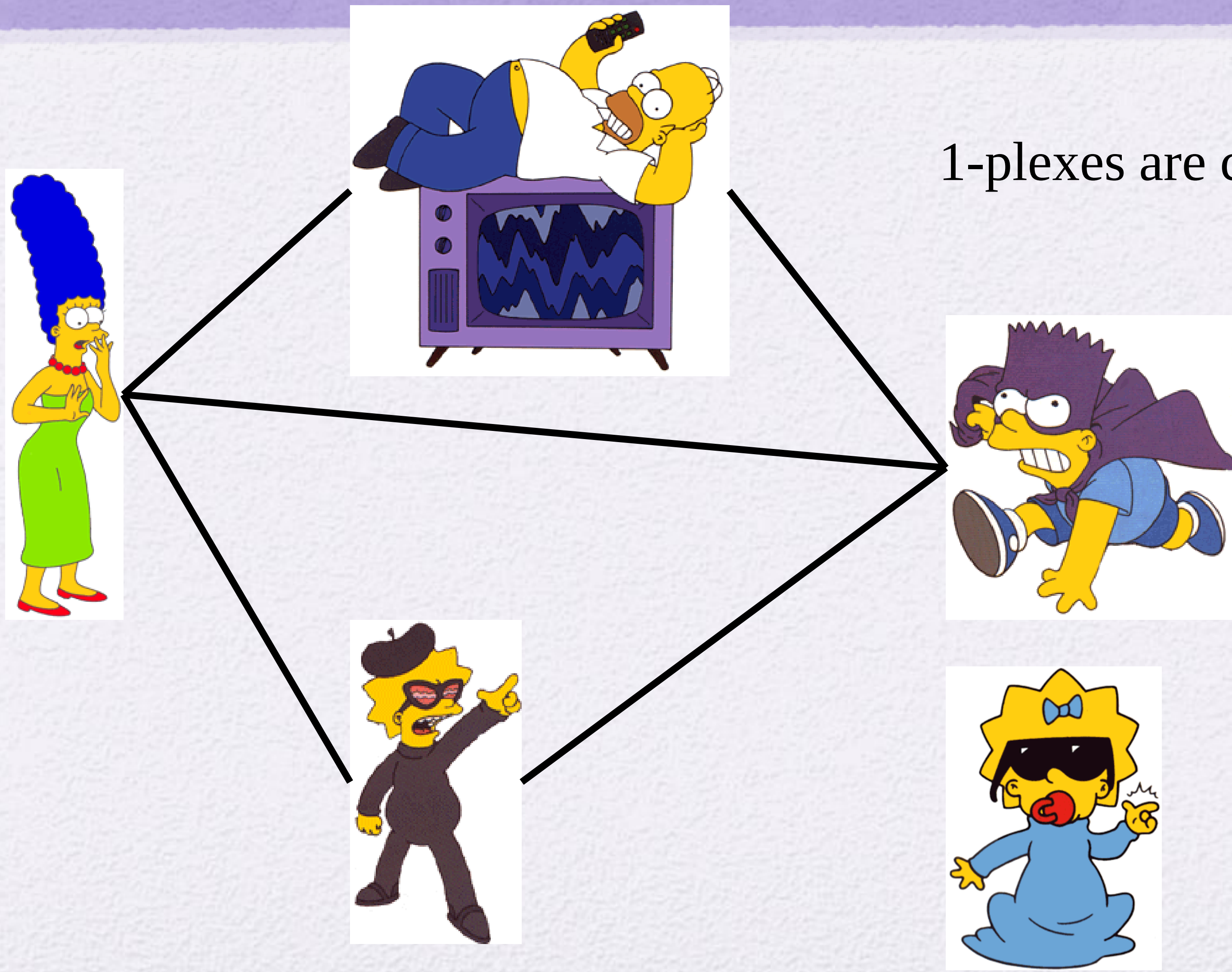
Different versions of cohesiveness

- Relax distance requirement between members
  - *k*-clique (Luce 1950)
  - *k*-club (Alba 1973)
- Relax the familiarity (# of neighbors) between members
  - *k*-plex (Siedman & Foster 1978)
  - *k*-core (Siedman 1983)

k-plexes

- Given a graph $G=(V, E)$ and some integer $k > 0$, a set $S \subseteq V$ is called a *k*-plex if every node of S has at most k-1 non-neighbors in S
- Cliques are 1-plexes
- NP-hard to find maximum *k*-plex, $\omega_k(G)$, in a graph G

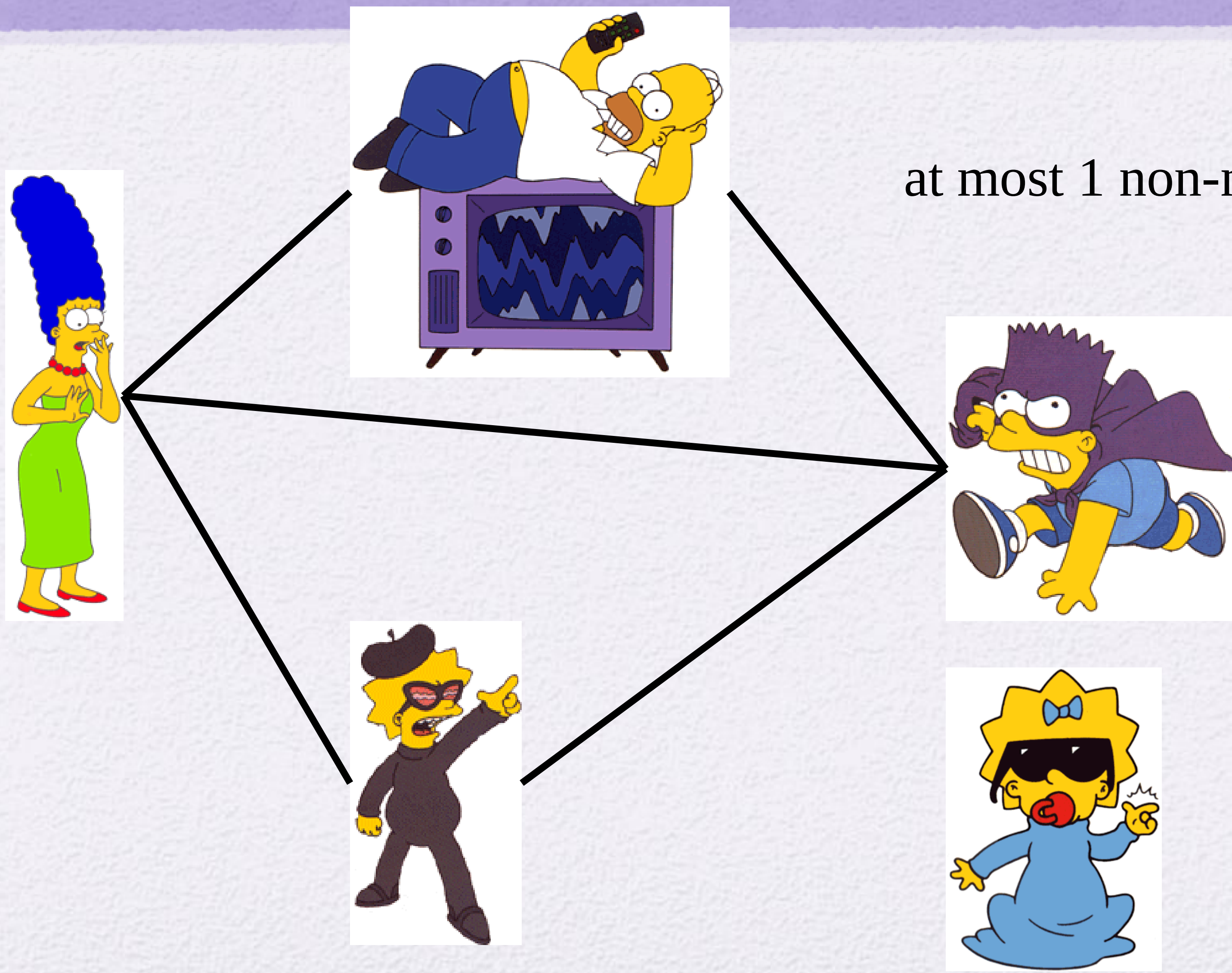
1-plexes



1-plexes are cliques

2-plexes

at most 1 non-neighbor



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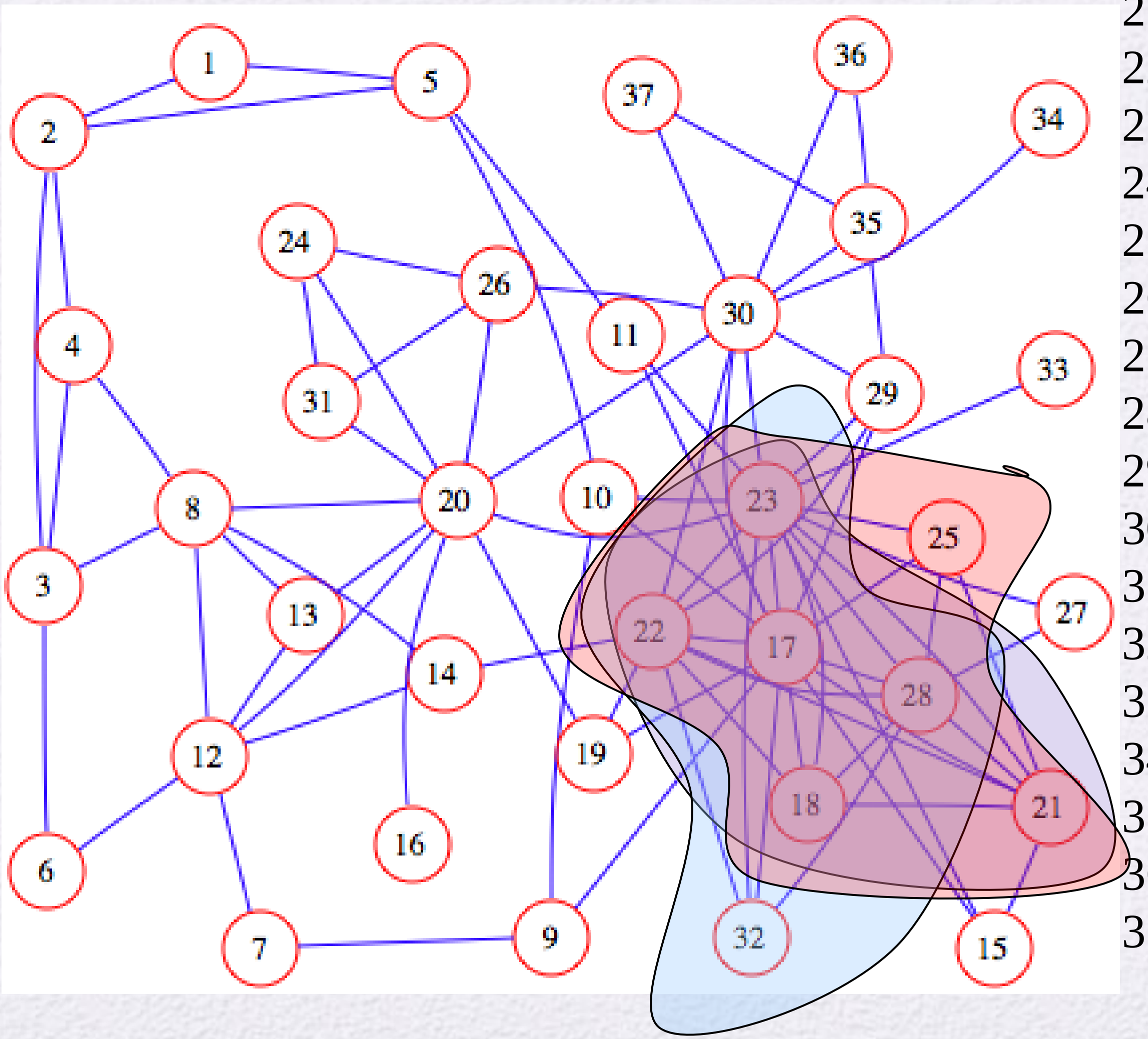

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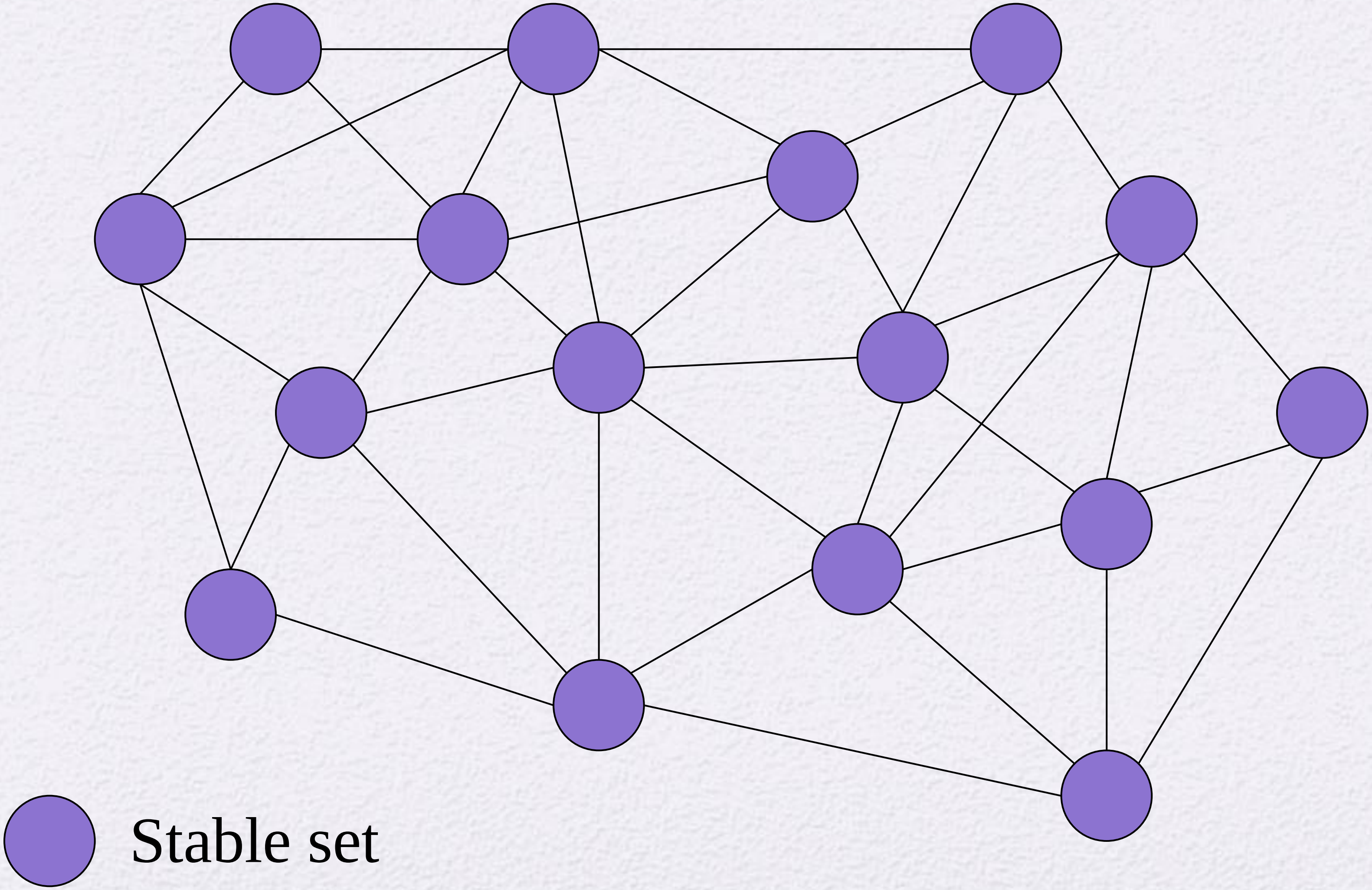
37) Alhazmi

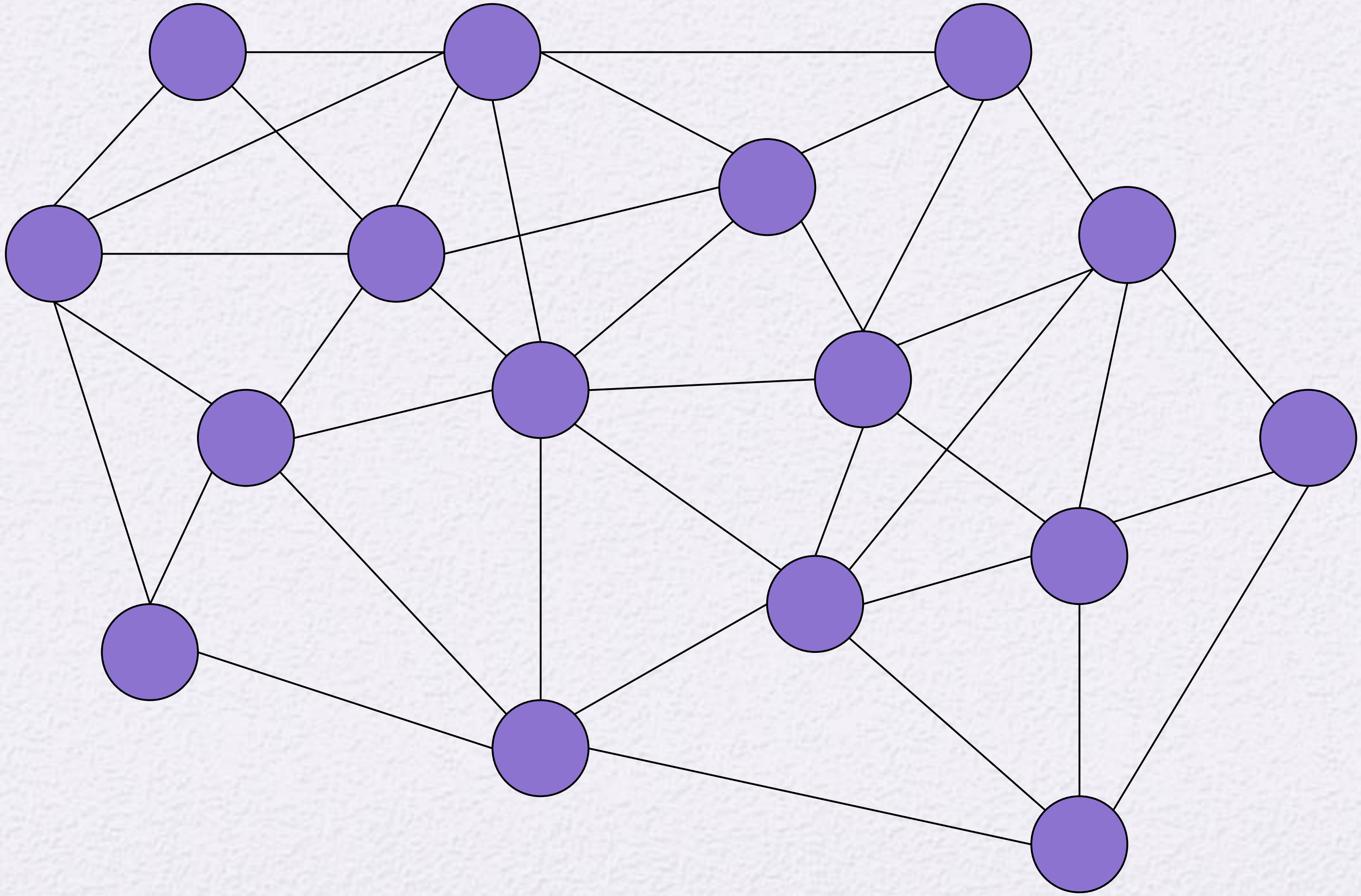
Ready for Co-*k*-plexes!!!

Another Example: Retail Location

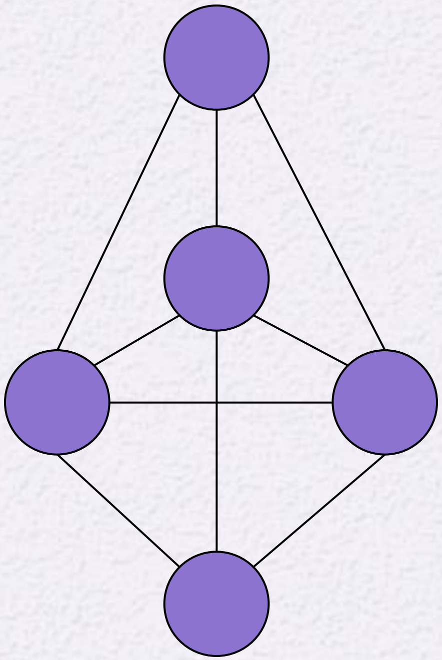


Starbucks in Springfield

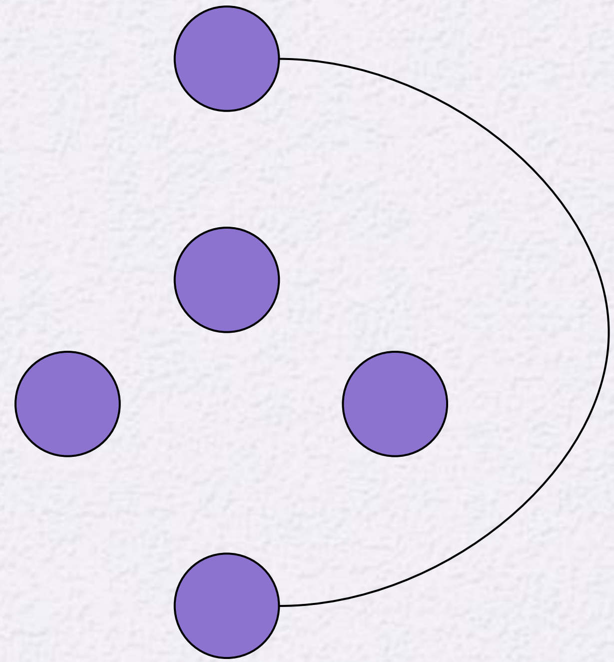
Another Example: Retail Location



k-plexes and co-*k*-plexes



G

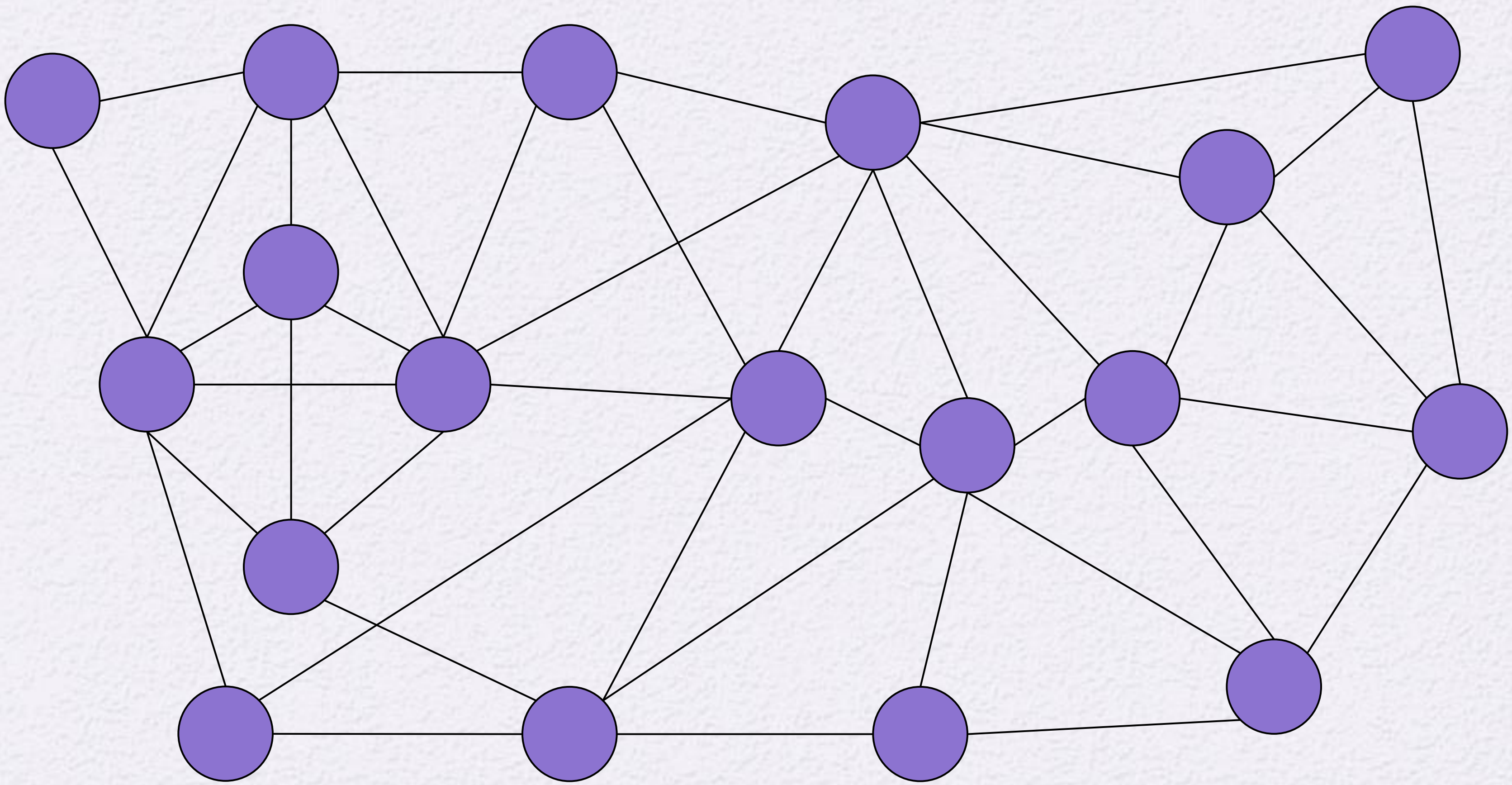


G^C

My Research: Combinatorial Optimization

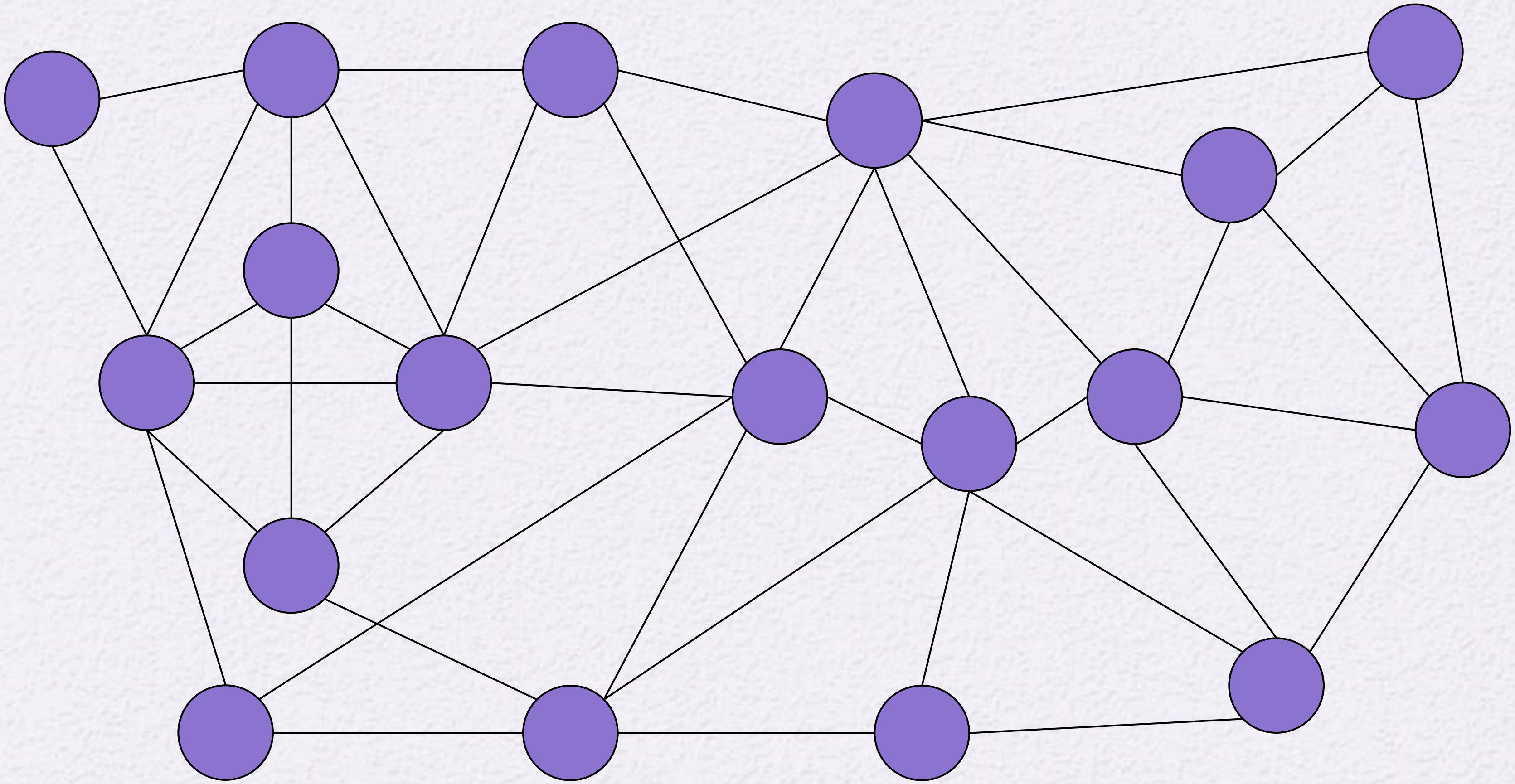
- How can we find the largest k-plex in a graph?
- Two ways I attack problems
  - Combinatorial (graph) algorithms
  - Polyhedral Combinatorics

Graph Coloring



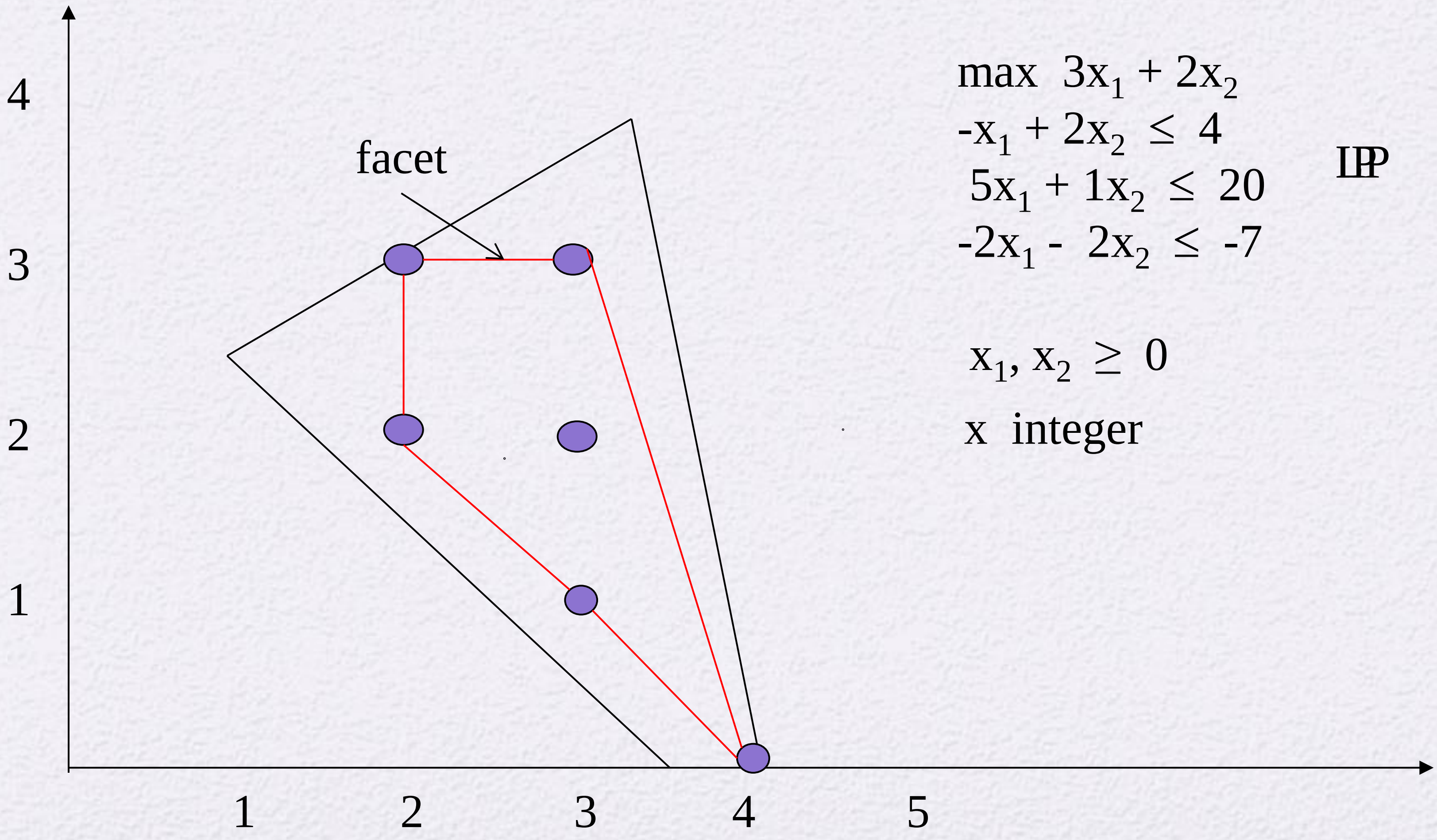
$\omega(G) \leq \chi(G)$

Co-*k*-plex Coloring



$$\omega_k(G) \leq \chi_k(G)$$

Linear and Integer Programming



$$\max\ 3x_1 + 2x_2$$
$$-x_1 + 2x_2 \le 4$$
$$5x_1 + 1x_2 \le 20$$
$$-2x_1 - 2x_2 \le -7$$

$$x_1, x_2 \ge 0$$

x integer

LP
IP

Wrap-Up

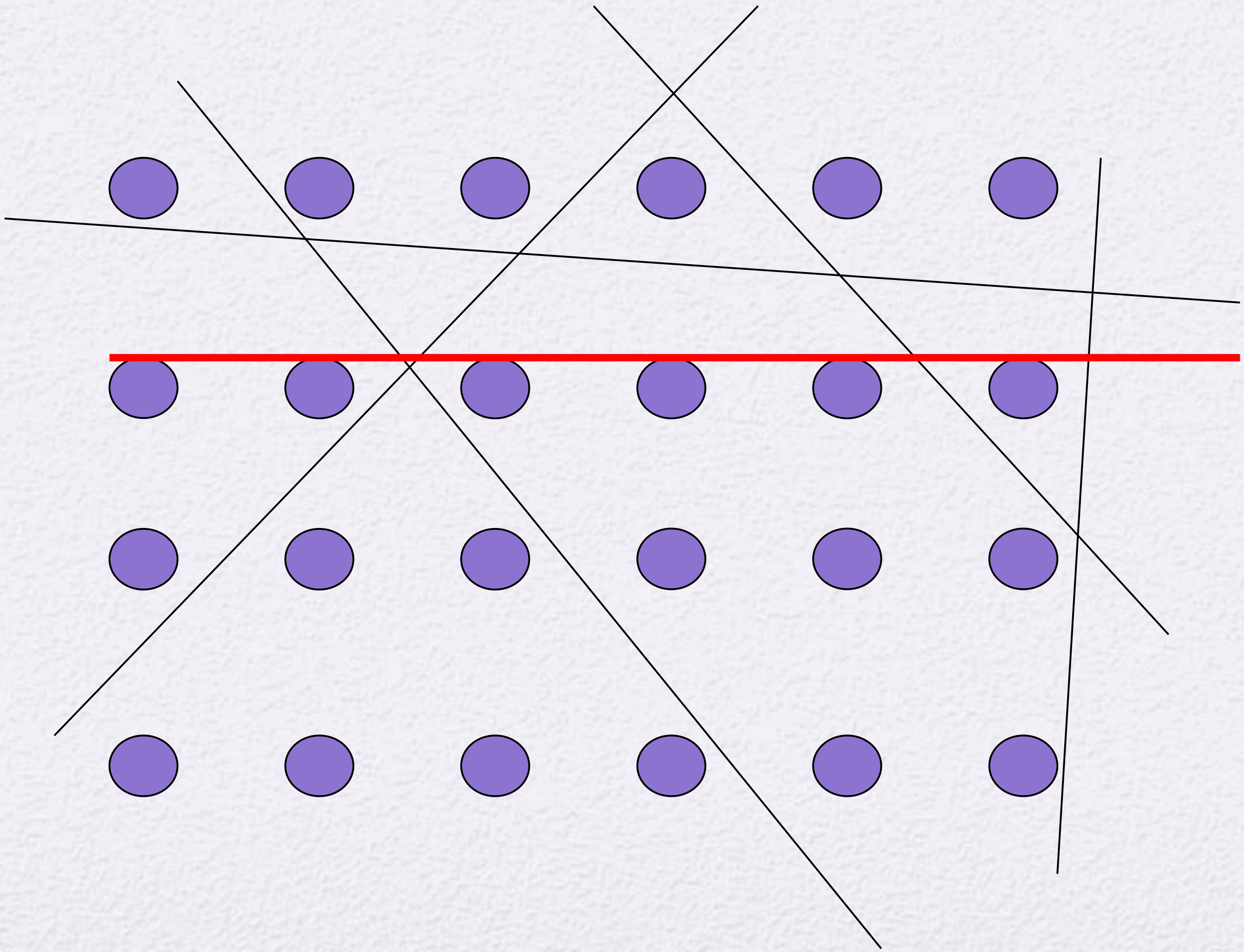
- Graph Definitions
- Applications
- Dominating Sets, TSP, Cliques & *k*-plexes

Polyhedral Approach

- Let N[v] denote the closed neighborhood of vertex v
- Let d(v) denote $|V \setminus N[v]|$

$$\text{Max} \sum_{v \in V} x_v$$
$$st.$$
$$\sum_{u \in V \setminus \mathrm{N}[v]} x_u \leq (k - 1)x_v + \mathrm{d}(v)(1 - x_v) \ \forall v \in V$$
$$x_v \in \{0, 1\} \ \forall v \in V$$

Polyhedral Approach



Acknowledgments

- My collaborator: Ben McClosky, Ph. D.
- NSF
  - DMI 0521209
  - DMS 0611723
  - CMMI 0926618

Any Questions?

Relevant Literature

- Seidman & Foster (1978)
  - Introduced *k*-plexes in context of social network analysis
- Balasundaram, Butenko, Hicks, and Sachdeva (2006)
  - IP formulation for maximum *k*-plex problem
  - NP-complete complexity result
- McClosky & Hicks (2007)
  - Co-2-plex polytope
- McClosky & Hicks (2008)
  - Graph algorithm to compute k-plexes

Co-k-plexes

- Given a graph $G=(V, E)$, a set $S \subseteq V$ is called a co-k-plex if $\Delta(G[S]) \leq k - 1$, where Δ denotes maximum degree
- Stable sets are co-1-plexes and co-k-plexes form independence systems
- NP-hard to find maximum co-k-plex, $\alpha_k(G)$ in a graph G
- Co-2-plexes correspond to vertex induced subgraphs of isolated nodes and matched pairs

Co-k-plex Polytope

- Given graph G, let $\mathscr{I}^k$ be the set of co-k-plexes in G
- For all $S \in \mathscr{I}^k$, let x^S be the incidence vector for S.
- Define $P_k(G) = \text{conv}(\{x^S : S \in \mathscr{I}^k\})$
- $P_2(G)$ shares many properties with $P_1(G)$

Co-2-plex analogs

- Padberg (1973)
  - Clique and odd hole inequalities
- Trotter (1975)
  - Web inequalities
- Minty (1980)
  - claw-free graphs

2-plex Inequalities

- Theorem (Padberg): If K is a maximal clique in G, then $\sum_{v \in K} x_v \leq 1$ is a facet for $P_1(G)$.

- Theorem (M & H, B et al.): If K is a maximal 2-plex in G such that $|K| > 2$, then $\sum_{v \in K} x_v \leq 2$ is a facet for $P_2(G)$

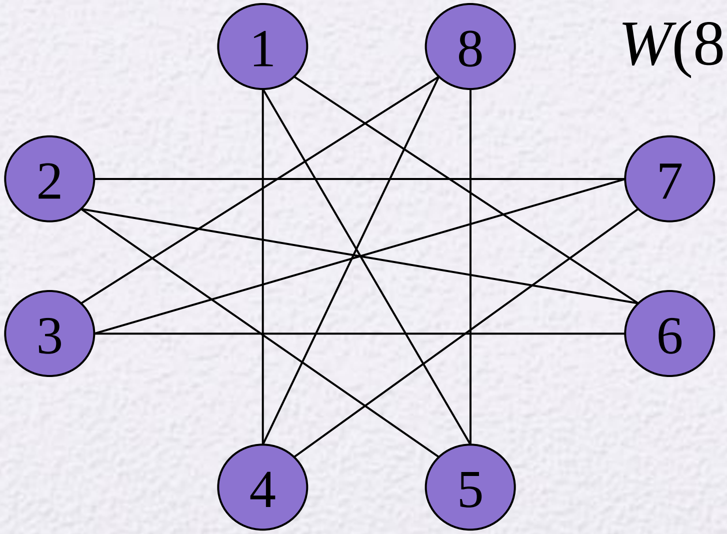
Odd-mod Hole Inequalities

- Theorem (Padberg): If *C* is an *n*-chordless cycle such that $n > 3$ is odd, then $\sum_{v \in V(C)} x_v \leq \lfloor n/2 \rfloor$ is a facet for $P_1(C)$.

- Theorem (M & H): If C is an *n*-chordless cycle such that $n > 2$ and $n \neq 0 \bmod 3$, then $\sum_{v \in V(C)} x_v \leq \lfloor 2n/3 \rfloor$ is a facet for $P_2(C)$

Webs

- For fixed integers $n \geq 1$and p such that $1 \leq p \leq \lfloor n/2 \rfloor$, the web $W(n, p)$ has n vertices and edges $E=\{(i, j): j=i+p, \ldots, i+n-p; \forall \text{ vertices } i\}$



$W(8, 3)$

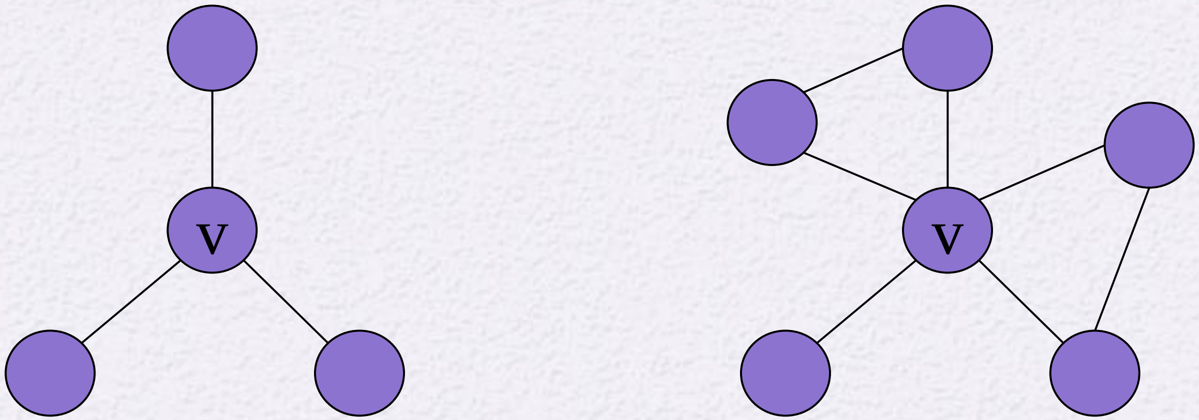
Web Inequalities

- Theorem (Trotter): If gcd(n, p) = 1, then $\sum_{v \in V(W(n,p))} x_v \leq p$ is a facet for $P_1(W(n, p))$.

- Theorem (M & H): If gcd(n, p + 1) = 1, then $\sum_{v \in V(W(n,p))} x_v \leq p + 1$ is a facet for $P_2(W(n, p))$.

k-claws

- Given an integer $k \geq 1$, the graph G is a *k*-claw if there exists a vertex *v* of G such that $V(G)=N[v]$, $N(v)$ is a co-*k*-plex, and $|N(v)| \geq \max\{3, k\}$



2-claw free graphs

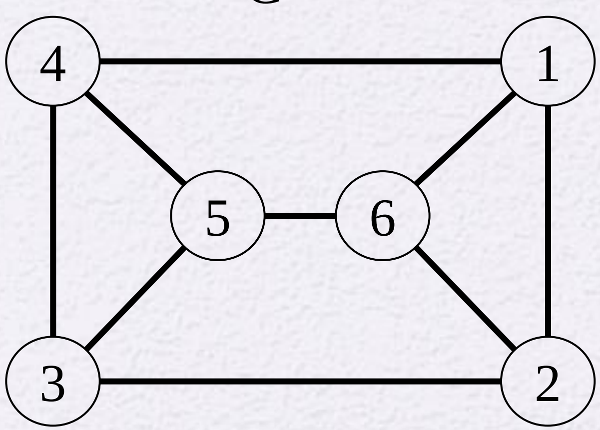
- Theorem (B & H): A graph G is 2-claw free if and only if $\Delta(G) \leq 2$ or G is 2-plex.

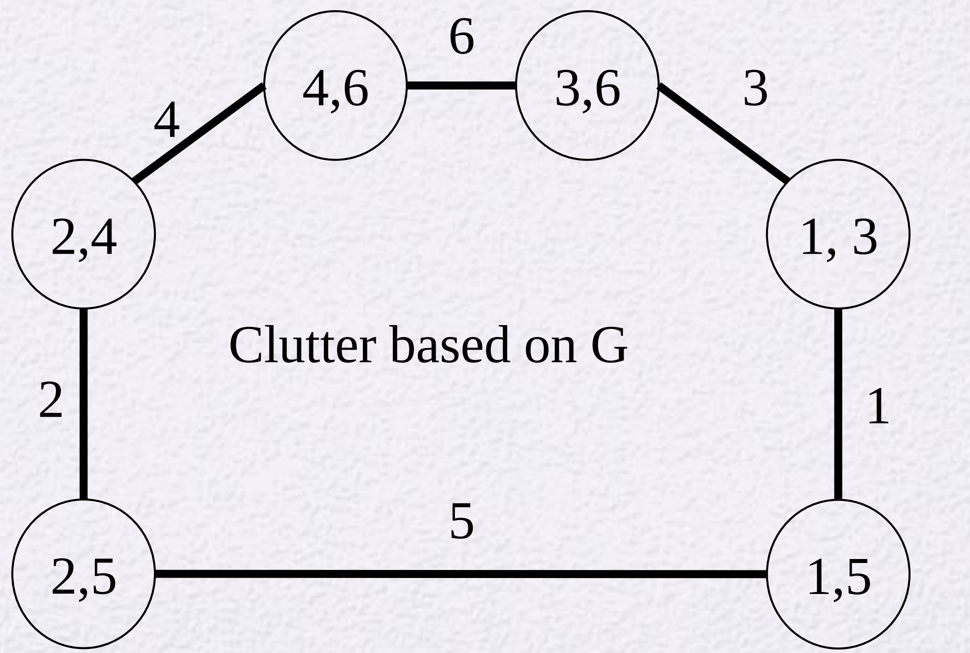
- This theorem will be used to describe a class of 0-1matrices A for which the polytope $P=\{x \in R^n_+: Ax \leq 2, x \leq 1\}$ is integral.

Clutters

- A clutter is a pair (V, E) where V is a finite set and E is a family of subsets of V none of which is included in another.

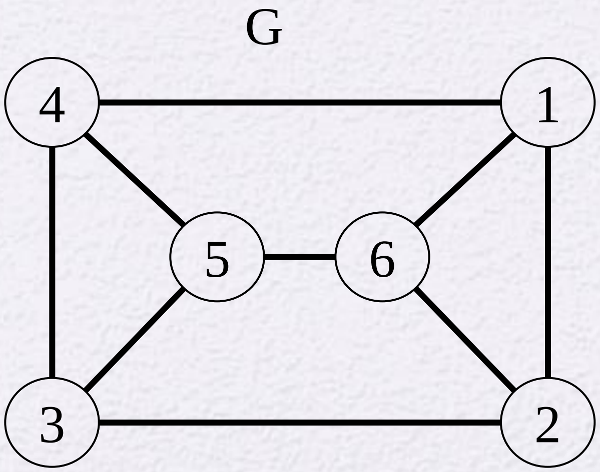
G

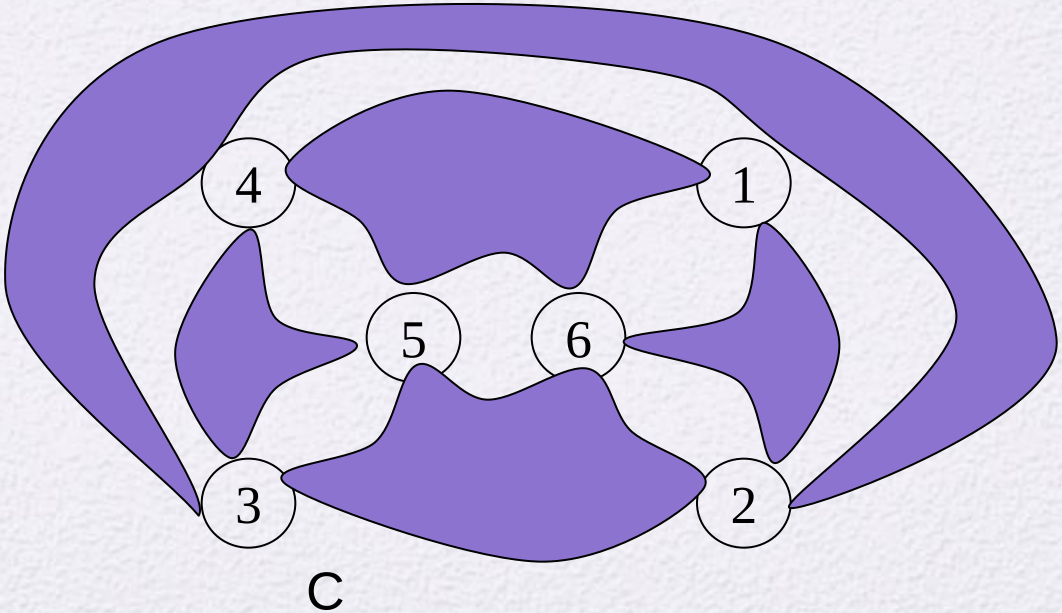




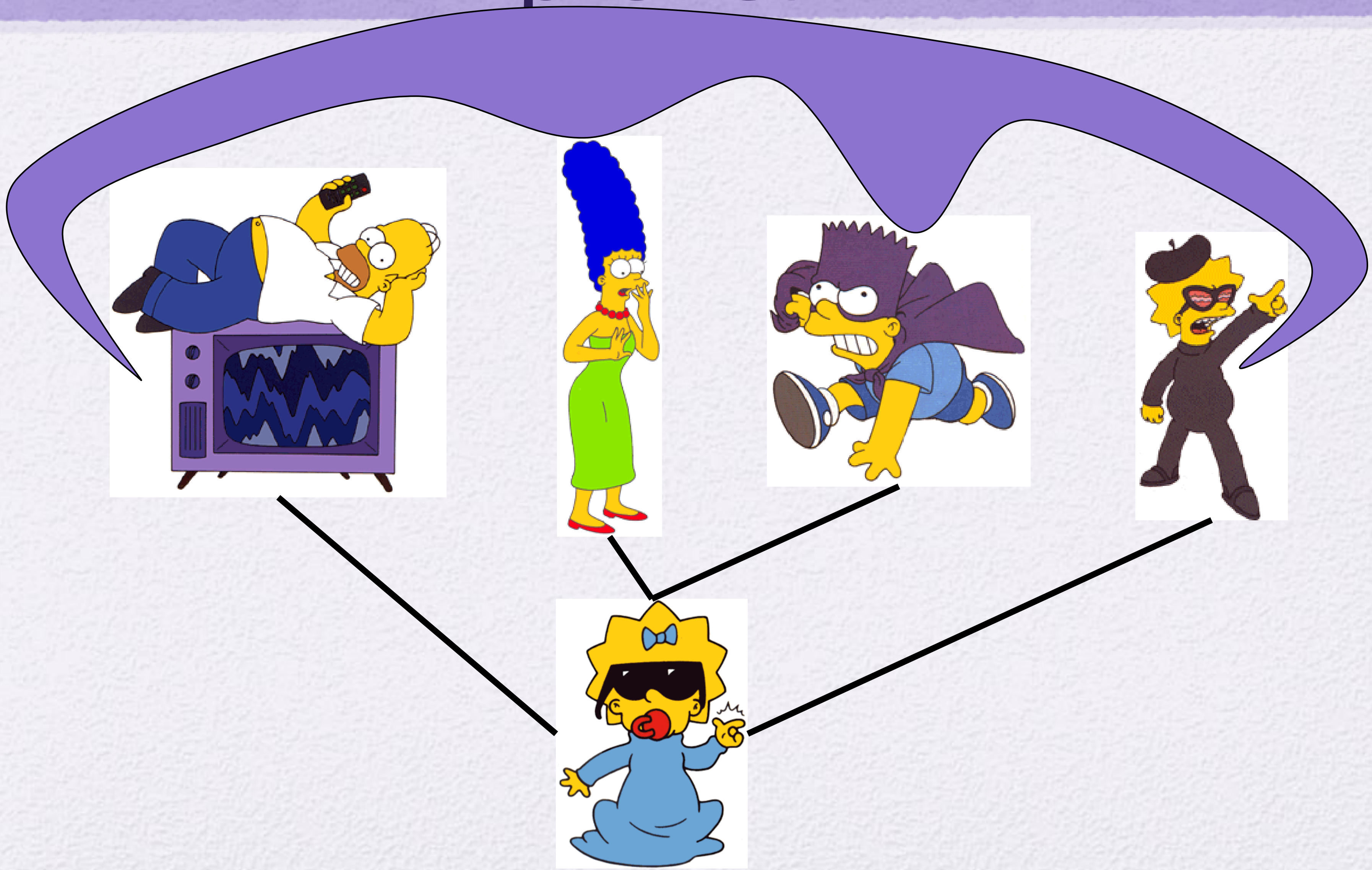
Clutters of Maximal 2-plexes

- Given a graph G, let C be the clutter whose vertices are V(G) and whose edges are maximal 2-plexes of G.





Clutters of Maximal 2-plexes



2-plex Clutter Matrices

Let A be the edge-vertex incidence matrix of C.

- Theorem (M & H): Let A be the 2-plex clutter matrix of G. The polytope $P=\{x \in R^n_+ : Ax \leq 2, x \leq 1\}$ is integral if and only if the components of G are 2-plexes, co-2-plexes, paths, or 0 mod 3 chordless cycles.

- Corrollary (M & H): Given a 2-plex clutter matrix A, there is a polynomial-time algorithm to determine if $P=\{x \in R^n_+ : Ax \leq 2, x \leq 1\}$ is integral.

Future Work

- Combinatorial algorithm to compute maximum k-plexes (involves *k*-plex coloring)
- Find facets of $P_k(G)$ for $k > 2$.
- Can k-plex clutter matrices give insight in polyhedra defined as $P=\{x \in R^n_+ : Ax \leq k, x \leq 1\}$?

Other inequalities

- Stable Sets
  - $\sum_{v \in I} x_v \leq k \;\; \forall$ stable sets I s.t. $|I| \geq k+1$
- Holes
  - $\sum_{v \in H} x_v \leq k+1 \;\; \forall$ holes H s.t. $|H| \geq k+3$
- Co-k-plexes
  - $\sum_{v \in S} x_v \leq \omega_k(S) \;\; \forall$ co-k-plexes S

2-plex Computational Results

G	n	m	density	ω(G)	BIS	UB	Time (sec)
c.200.1	200	1534	.077	12	12	12	57.3
c.200.2	200	3235	.163	24	24	24	46.9
c.200.5	200	8473	.426	58	58	58	40.2
h.6.2	64	1824	.905	32	32	32	.47
h.6.4	64	704	.349	4	6	6	4.4
h.8.2	256	31616	.969	128	128	130	>86000
h.8.4	256	20864	.639	16	16	46	>86000
j.8.2.4	28	210	.556	4	5	5	3.6
j.8.4.4	70	1855	.768	14	14	14	7424
j.16.2.4	120	5460	.765	8	10	14	>86000
k.4	171	9435	.649	11	15	26	>86000
m.a9	45	918	.927	16	26	26	2.3

Pop Quiz: Question #1

Who was the first African-American to receive a PhD in Mathematics?

Elbert F. Cox

Dissertation: Polynomial Solutions of Difference Equations
Ph.D. Cornell University, 1925
Advisor: William Lloyd Garrison

Pop Quiz: Question #2

Who was the first African-American to receive a PhD in Mathematics at Rice University?

Raymond Johnson

Dissertation: A Priori Estimates and Unique Continuation Theorems for Second Order Parabolic Equations
Ph.D. Rice University, 1969
Advisor: Jim Douglass Jr.