TRIGONOMETRIC PROOFS OF CONGRUENT TRIANGLES

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Types of proofs

Paragraph Proof – This proof consists of a detailed paragraph explaining the proof process. The paragraph contains steps and supporting justifications which prove the statement true.

Two column proof – This proof consists of two columns, where the first column contains a numbered chronological list of steps, called *Statements,* leading to the desired conclusion. The second column contains the justifications, called *Reasons*, to support each step in the proof.

Flow Proof (Chart Proof) - This proof format shows the structure of a proof using boxes and connecting arrows. The appearance is like a detailed drawing of the proof or perhaps a graphical organizer.

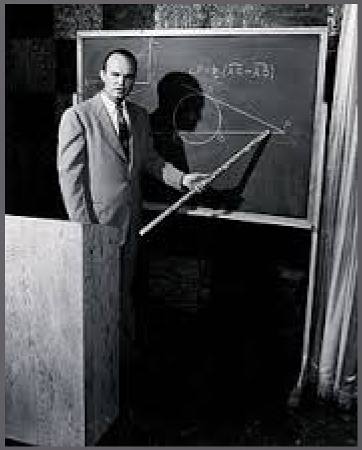
Trends in Proofs

- Traditional Deductive Proofs

- Demonstrating a Theorem
(numerical/algebraic calculations or measurements)

- Hands-On Activities
(with some materials and manipulatives)

- Audio/Visual Materials
(traditional lecturing or cartoons)



STATEMENTS ABOUT CONGRUENT TRIANGLES

Book	SAS	ASA	AAS	SSS	HL
McDougal Littell (Ron Larsen)	Postulate	Postulate	Theorem	Postulate	Theorem
Big Ideas, LLS (Ron Larsen)	Theorem (proved)	Theorem (proved)	Theorem (proved)	Theorem (proved)	Theorem (proved)
HOLT	Postulate	Theorem	Theorem		
Pearson	Postulate	Postulate	Theorem	Postulate	Theorem (proof in footnote)
Atanasian	Theorem (proved)	Theorem (proved)	Theorem (proved)	Theorem (proved)	Theorem (proved)
Euclid	Theorem	Theorem		Theorem	Theorem
Seymur	Theorem (proved)	Theorem (proved)	Theorem (proved)	Theorem (proved)	Theorem (proved)
Michael Serra	Conjecture	Conjecture	Conjecture	Conjecture	Conjecture

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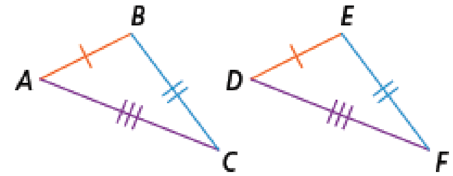
p. 153

take note

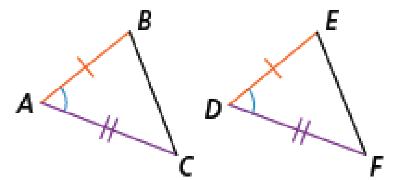
Postulate 4-1 Side-Side-Side (SSS) Postulate

Postulate	If . . .	Then . . .
If the three sides of one triangle are congruent to the three sides of another triangle, then the two triangles are congruent.	$\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{AC} \cong \overline{DF}$	$\triangle ABC \cong \triangle DEF$



Postulate 4-2 Side-Angle-Side (SAS) Postulate

Postulate	If . . .	Then . . .
If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the two triangles are congruent.	$\overline{AB} \cong \overline{DE}$, $\angle A \cong \angle D$, $\overline{AC} \cong \overline{DF}$	$\triangle ABC \cong \triangle DEF$

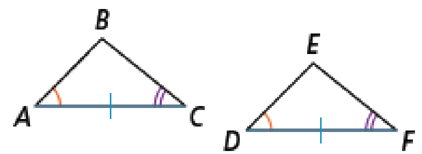


p. 158

take note

Postulate 4-3 Angle-Side-Angle (ASA) Postulate

Postulate	If . . .	Then . . .
If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the two triangles are congruent.	$\angle A \cong \angle D$, $\overline{AC} \cong \overline{DF}$, $\angle C \cong \angle F$	$\triangle ABC \cong \triangle DEF$



p. 174

take note

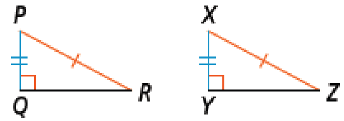
Theorem 4-6 Hypotenuse-Leg (HL) Theorem

Theorem

If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and a leg of another right triangle, then the triangles are congruent.

If . . .

$\triangle PQR$ and $\triangle XYZ$ are right △s, $\overline{PR} \cong \overline{XZ}$, and $\overline{PQ} \cong \overline{XY}$



Then . . .

$\triangle PQR \cong \triangle XYZ$

For a proof of Theorem 4-6, see the Reference section on page 683.

p. 685

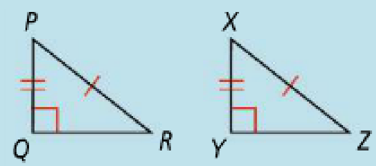
Theorem 4-6

Hypotenuse-Leg (HL) Theorem

If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and a leg of another right triangle, then the triangles are congruent. (p. 174)

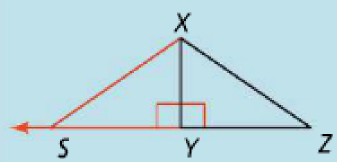
Proof

Given: $\triangle PQR$ and $\triangle XYZ$ are right triangles, with right angles Q and Y. $\overline{PR} \cong \overline{XZ}$ and $\overline{PQ} \cong \overline{XY}$.



Prove: $\triangle PQR \cong \triangle XYZ$

Proof: On $\triangle XYZ$, draw $\overrightarrow{ZY}$.



Mark point S so that $YS = QR$. Then, $\triangle PQR \cong \triangle XYS$ by SAS.

Since corresponding parts of congruent triangles are congruent, $\overline{PR} \cong \overline{XS}$. It is given that $\overline{PR} \cong \overline{XZ}$, so $\overline{XS} \cong \overline{XZ}$ by the Transitive Property of Congruence. By the Isosceles Triangle Theorem, $\angle S \cong \angle Z$, so $\triangle XYS \cong \triangle XYZ$ by AAS. Therefore, $\triangle PQR \cong \triangle XYZ$ by the Transitive Property of Congruence.

PROOF AS A SEQUENCE OF ALGEBRAIC/TRIGONOMETRIC CALCULATIONS

Prerequisites:

- Cosine Theorem (Law of Cosines)
- Sine Theorem (Law of Sines)
- Interior Angles Theorem

DEFINITION OF CONGRUENCY

A triangle consists of 3 sides and 3 angles

$$< s_1, s_2, s_3, s_4, s_5, s_6 > = < a, b, c, \angle A, \angle B, \angle C >$$

Algebraically, a triangle is term of six numbers (elements):

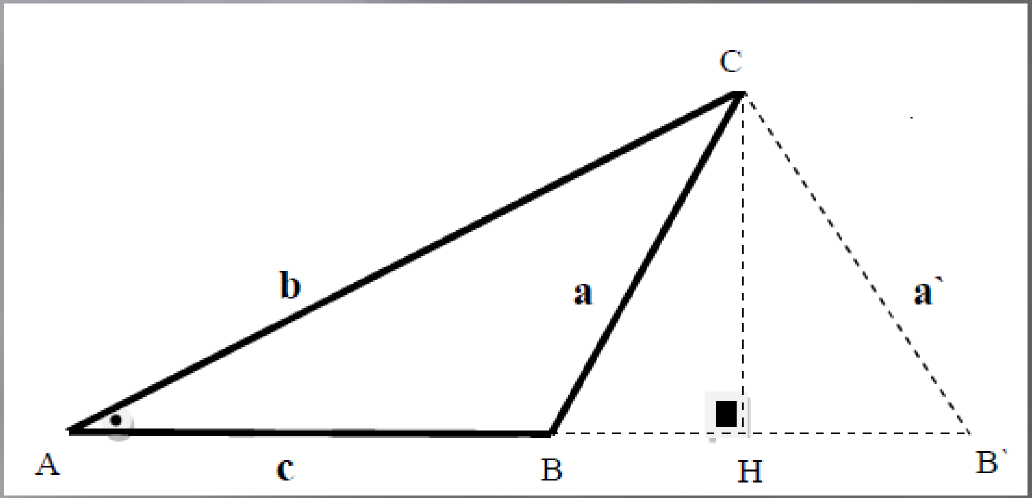
$$< s_1, s_2, s_3, s_4, s_5, s_6 >$$

DEFINITION:
If for any given three elements, it is possible to find some unique formulae for the other three elements, then all triangles with the same given data are congruent.

$$\begin{aligned} s_{i_1} &= f(s_{i_4}, s_{i_5}, s_{i_6}) \\ s_{i_2} &= \phi(s_{i_4}, s_{i_5}, s_{i_6}) \\ s_{i_3} &= \psi(s_{i_4}, s_{i_5}, s_{i_6}) \end{aligned}$$

SAS THEOREM

Given: sides b, c and the angle ∠A
Find: side a, angle ∠B, angle ∠C



Find:

$$a = f(b, c, \angle A)$$
$$\angle B = \phi(b, c, \angle A)$$
$$\angle C = \psi(b, c, \angle A)$$

SAS THEOREM (CONT'D)

Process:

$$a^2 = b^2 + c^2 - 2bc \cdot cos(A)$$

$$\frac{a}{sin(A)} = \frac{b}{sin(B)}$$

$$m\angle A + m\angle B + m\angle C = \pi$$

Solution:

If $b \cdot cos\ A > c$, then $\angle B$ is acute and $m\angle B = \arcsin(\frac{b}{c} \cdot sinA)$

If $b \cdot cos\ A < c$, then $\angle B$ is obtuse and $m\angle B = \pi - \arcsin(\frac{b}{c} \cdot sinA)$

If $b \cdot cos\ A = c$, then $\angle B$ is a right angle and $m\angle B = 90^\circ$

SAS THEOREM (END)

Final answer

$$a = \sqrt{b^2 + c^2 - 2bc \cdot cosA}$$

$$m\angle B = \arcsin(\frac{b}{c} \cdot sinA) \quad \text{- if } b \cdot cos\ A > c$$

$$m\angle B = \pi - \arcsin(\frac{b}{c} \cdot sinA) \quad \text{- if } b \cdot cos\ A < c$$

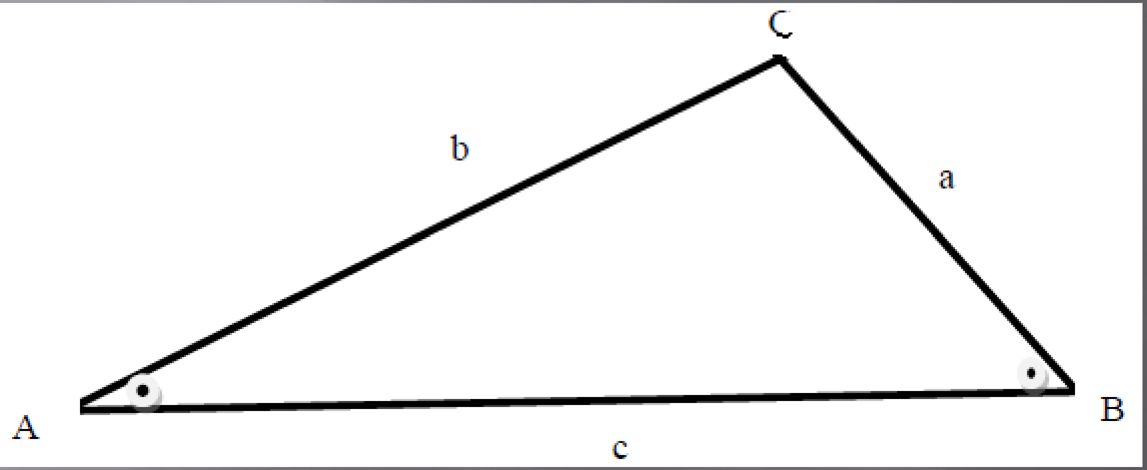
$$m\angle B = \frac{\pi}{2} \quad \text{- if } b \cdot cos\ A = c$$

$$m\angle C = \pi - m\angle A - m\angle B$$

END OF THEOREM

ASA THEOREM

Given: side c, m∠A, m∠B
Find: side b, side a, m∠C



Find:

$$a = f(c, m\angle A, m\angle B)$$
$$b = f(c, m\angle A, m\angle B))$$
$$m\angle C = \psi(c, m\angle A, m\angle B)$$

ASA THEOREM (CONT'D)

Process:

$$a^2 = b^2 + c^2 - 2bc \cdot cos(A)$$
$$\frac{a}{sin(A)} = \frac{b}{sin(B)}$$
$$m\angle A + m\angle B + m\angle C = \pi$$

Solution

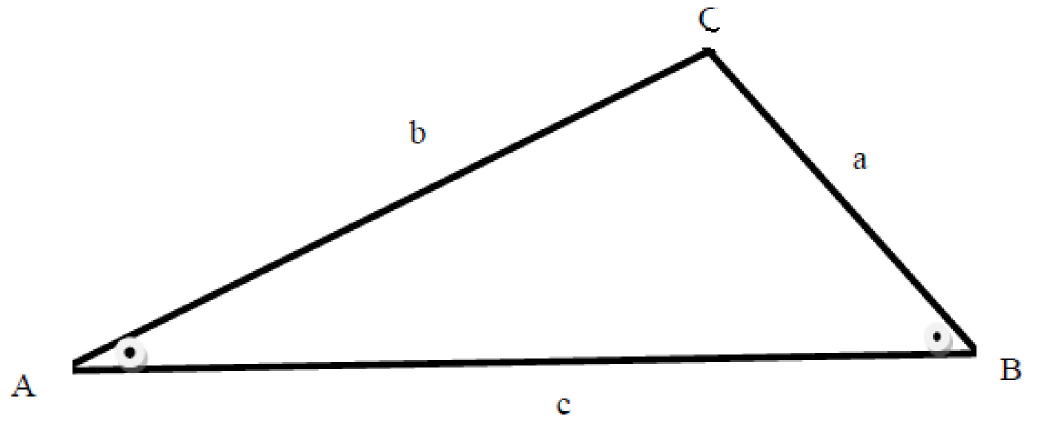
$$m\angle C = \pi - m\angle A - m\angle B$$
$$a = \frac{c \cdot sin(A)}{sin(\pi - m\angle A - m\angle B)}$$
$$b = \frac{c \cdot sin(B)}{sin(\pi - m\angle A - m\angle B)}$$

End of Theorem

SSS THEOREM

Given: side a, side b, and side c

Find: angle ∠A, angle ∠B, and ∠C



Find

$$m\angle A = f(a, b, c)$$
$$m\angle B = f(a, b, c)$$
$$m\angle C = f(a, b, c)$$

SSS THEOREM

Process

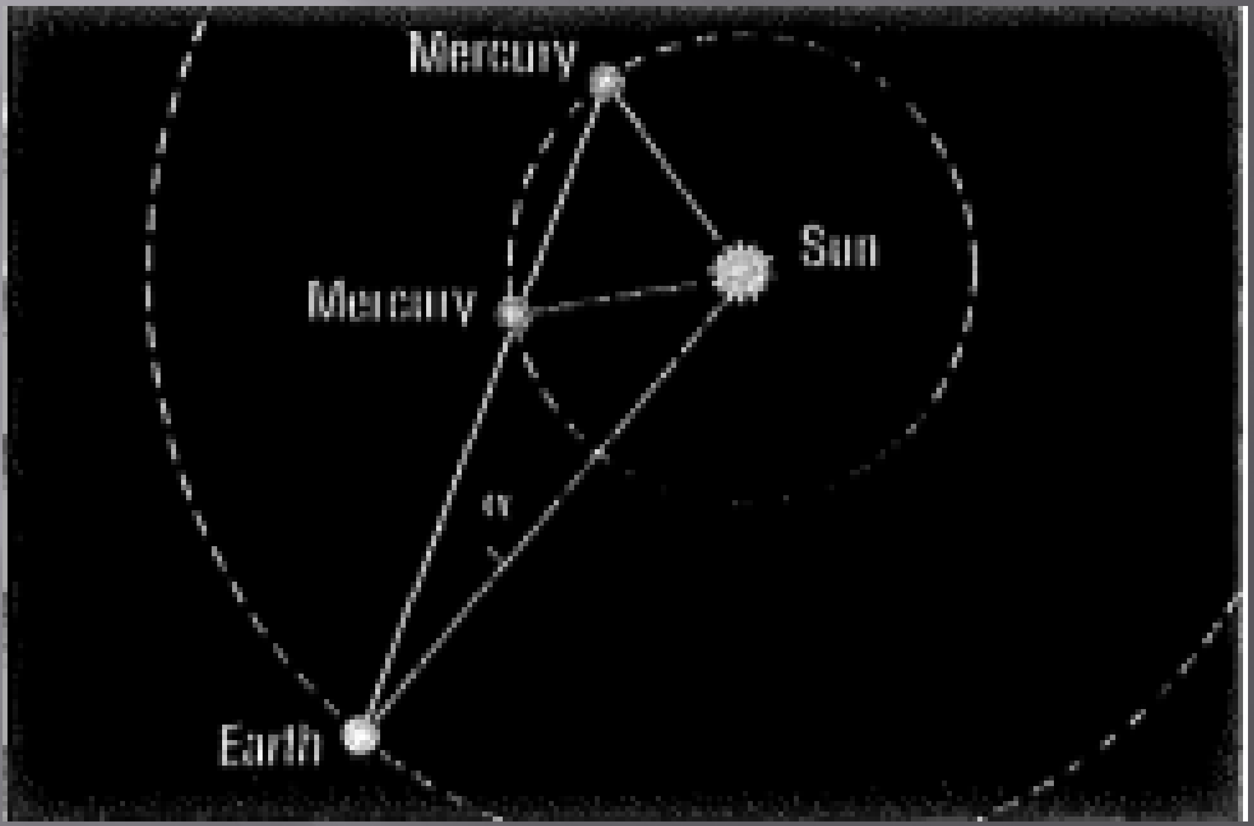
$$a^2 = b^2 + c^2 - 2ab \cdot cos(A)$$
$$\frac{b}{m\angle B} = \frac{a}{m\angle A}$$
$$m\angle A + m\angle C + m\angle C = \pi$$

Solution

$$m\angle A = arccos(\frac{b^2+c^2-a^2}{2ab})$$
$$m\angle B = arcsin(\frac{b}{a}sin(A))$$
$$m\angle C = \pi - arccos(\frac{b^2+c^2-a^2}{2ab}) - arcsin(\frac{b}{a}sin(A))$$

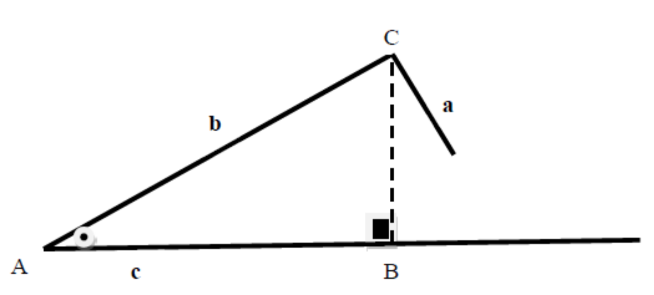
End of Theorem

SSA – THE CASE OF AMBIGUITY



SSA Theorem

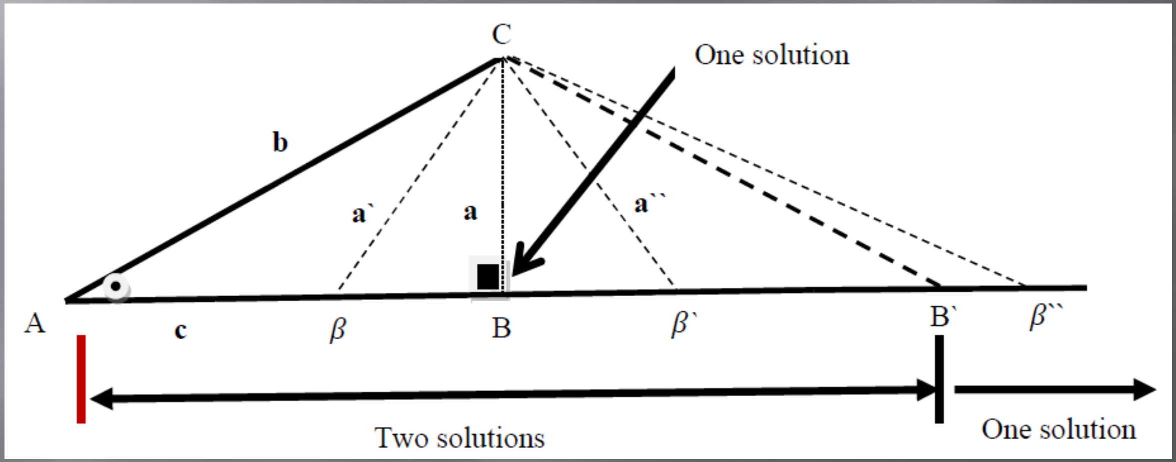
Given : side a, side b, angle <A (is not between sides a and b)
Find: angle ∠B, angle ∠C, side C



No solution if $b \cdot sinA >$ a

Two solutions if $b \cdot sinA < a < b$

One solution if $b \cdot sinA = a$ or $b < a$



SSA Theorem (con'd)

Process:

$$\frac{a}{sinA} = \frac{b}{\sin B}$$

$$m\angle A + m\angle B + m\angle C = \pi$$

$$c = \sqrt{a^2 + b^2 - 2ab \cdot cosC}$$

Solution:

$$m\angle B = \arcsin(\frac{b}{a} \cdot sinA), \quad \text{if } \angle\text{B is acute}$$

$$m\angle B = \pi - \arcsin(\frac{b}{a} \cdot sinA), \text{ if } \angle\text{B is obtuse}$$

$$m\angle C = \pi - m\angle A - m\angle B$$

$$c = \sqrt{a^2 + b^2 - 2ab \cdot cosC}$$

END OF THEOREM

CONCLUSIONS

- Statements about congruent triangles (SAS, ASA, SSS, HL) are THEOREMS, not postulates or conjectures.
- Proofs of congruent triangles can be implemented into Trigonometry courses.
- Other geometric theorems could be proved as well with algebraic and trigonometric methods.

THANK YOU!

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