The Language of Mathematics

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Outline of Talk

1. Language of Mathematics
2. Propositional Calculus, zeroth-order logic
3. Predicate Calculus, first-order logic
4. Applications to Mathematics
5. Institute for Mathematics and Computer Sciences (IMACS)

Ingredients of Mathematics

1. Language
2. Boolean nature of sentences
3. Axioms of Mathematics
   (a) A *set* is a collection of objects called *elements*.
   (b) Algebra: For any real x, we have $x + 0 = x$.
   (c) Geometry: A line can be extended arbitrarily.

The Boolean nature of sentences consists of a function

$$V : \{\text{absolute sentences}\} \to \{\text{T,F}\}$$

V(“There are infinitely many primes.”) = T

V(“Every angle can be trisected.”) = F

V(Goldbach’s conjecture) = ???

V(“There is organic material on Neptune.”) = ???

V(“It is sinful to covet your neighbor’s wife.”) = ND

The Boolean nature of sentences does not apply to:

1. Interjections: “Eeeek!”
2. Interrogatives: “Don’t you want me, baby?”
3. Imperatives: “Don’t mess with Texas.”
4. Jussives: “May a plague strike your house.”

Constructing compound sentences

1. Negation: $\neg P$
2. Conjunction: $P \wedge Q$
3. Disjunction: $P \vee Q$
4. Implication: $P \Rightarrow Q$
5. Biconditional: $P \Leftrightarrow Q$

Translating Sentences

Anne is beautiful but I am not beautiful.

$$P \wedge Q$$

I will pass chemistry or I will pass mathematics.

$$P \vee Q$$

I will pass chemistry or I will pass mathematics, but not both.

$$(P \vee Q) \wedge \neg(P \wedge Q)$$

Either you will pass the course or you will fail the course.

$$P \vee \neg P$$

Translating implications and biconditionals

$P =$ "you are very lucky", $Q =$ "you will win the lottery"

You will win the lottery if you are very lucky.

$$P \Rightarrow Q$$

You will win the lottery only if you are very lucky.

$$Q \Rightarrow P$$

You will win the lottery if and only if you are very lucky.

$$P \Leftrightarrow Q$$

Truth table for basic binary operations

P	Q	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Provide the truth table for $(P \vee \neg Q) \Rightarrow P$

P	Q	$\neg Q$	$P \vee \neg Q$	$(P \vee \neg Q) \Rightarrow P$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	T	F

Tautologies and Contradictions

P	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
T	F	T	F
F	T	T	F

Yet another tautology

P	Q	$P \wedge Q$	$(P \wedge Q) \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Manufacturing Tautologies

P	Q	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

Result:

$$\neg\ (\ P\ \wedge\ Q\ ) \Leftrightarrow (\ \neg P\ \vee\ \neg Q\ )$$

is a tautology!

DeMorgan's Laws of Negation

$$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$$

$$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$$

$$\neg(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$$

The Contrapositive

P	Q	$P \Rightarrow Q$	$\neg Q \Rightarrow \neg P$	$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

The Limitations of Propositional Calculus

$P(x, y)$ means "x loves y."

$\neg P(x, y)$ means "x does not love y."

We can assert the following:

$$P(\text{Romeo, Juliet})$$
$$\neg P(\text{Tom Cruise, Nicole Kidman})$$
$$P(\text{Richard, Mickey Mouse})$$

Consider the sentence “Everyone loves me.”

P(Anne, me)	P(Joanna, me)	P(Richard, me)
P(Mickey, me)	P(Hillary Clinton, me)	P(LL Cool J, me)
P(Monica Lewinsky, me)	P(Roseanne, me)	P(Idi Amin, me)
P(Boy George, me)	P(Roy Orbison, me)	P(Connie Chung, me)
P(Kermit the Frog, me)	P(Dennis Rodman, me)	P(Stephen King, me)
...	...	...

Predicate Calculus, or First-Order Logic

The sentence "Everyone loves me" is symbolically:

$$\forall x \ [\ P(x, \text{me})\ ]$$

which reads "For all x, x loves me."

Better yet, if $\mathcal{U}$ is the set of all people, write

$$\forall x \in \mathcal{U} \quad [\ P(x, \text{me})\ ]$$

The symbol $\forall$ is called the *universal quantifier*.

Negating the Universal

We want the negation of $\forall x \in \mathcal{U} \ \ [\ P(x, \mathrm{me})\ ]$.

Attempt: $\forall x \in \mathcal{U} \ \ [\ \neg P(x, \mathrm{me})\ ]$

Translation: "No one loves me."

We want: "Someone does not love me" or "There exists someone who does not love me."

The Existential Quantifier

The sentence

“There exists someone who does not love me”

can be written

$$\exists x \in \mathcal{U} \ \ [\ \neg P(x, \mathrm{me})\ ]$$

The symbol $\exists$ is called the *existential quantifier*.

Some Tautologies

$$\neg\{\ \forall x \in \mathcal{U}\ \ [\ P(x, \mathrm{me})\ ]\ \} \Leftrightarrow \{\ \exists x \in \mathcal{U}\ \ [\ \neg P(x, \mathrm{me})\ ]\ \}$$

$\neg$ {Everyone loves me} $\Leftrightarrow$ {Someone does not love me}

$$\neg\{\ \exists x \in \mathcal{U}\ \ [\ P(x, \mathrm{me})\ ]\ \} \Leftrightarrow \{\ \forall x \in \mathcal{U}\ \ [\ \neg P(x, \mathrm{me})\ ]\ \}$$

$\neg$ {Someone loves me} $\Leftrightarrow$ {No one loves me}

{No one loves me} $\Rightarrow$ $\neg$ {Everyone loves me}

More Complicated Examples

Translate "Everyone is loved by someone."

Let x be the existential, and y the universal.

The ingredients are $\forall y \in \mathcal{U}$, $\exists x \in \mathcal{U}$, $P(x, y)$.

Which is right?

1. $\exists x \in \mathcal{U}\ \forall y \in \mathcal{U}\ \ [\ P(x, y)\ ]$.
2. $\forall y \in \mathcal{U}\ \exists x \in \mathcal{U}\ \ [\ P(x, y)\ ]$.

The Importance of Order

Option 1: $\exists x \in \mathcal{U}\ \forall y \in \mathcal{U}\ \ [\ P(x,y)\ ]$

This means: "Someone loves everyone."

Option 2: $\forall y \in \mathcal{U}\ \exists x \in \mathcal{U}\ \ [\ P(x,y)\ ]$

This means: "Everyone is loved by someone."

Negating First-Order Sentences

Find the negation of $\forall y \in \mathcal{U}\ \exists x \in \mathcal{U}\ \ [\ P(x,y)\ ]$.

Solution: $\neg \{\ \forall y \in \mathcal{U}\ \exists x \in \mathcal{U}\ \ [\ P(x,y)\ ]\ \}$

$$\Leftrightarrow \exists y \in \mathcal{U}\ \neg \{\ \exists x \in \mathcal{U}\ \ [\ P(x,y)\ ]\ \}$$

$$\Leftrightarrow \exists y \in \mathcal{U}\ \forall x \in \mathcal{U}\ [\neg\, P(x,y)\ ]$$

Translation: There is someone who is loved by no one.

A confused Glinda

1. “Are you a good witch or a bad witch?”
2. “Only bad witches are ugly.”
3. “I’m a little muddled.”

Question: Did Glinda insult Dorothy?

Reinterpret Glinda's sentence

Only bad witches are ugly.

Reinterpret: A witch is ugly only if s/he is bad.

Let C be the collection of all creatures,

W be the subset of witches,

B be the subset of bad creatures,

U be the subset of ugly creatures.

$$(\forall P \in C)[((P \in W) \wedge (P \in U)) \Rightarrow (P \in B)]$$

A Venn diagram

Negating Glinda's sentence

Only bad witches are ugly.

$(\forall P \in C)[((P \in W) \wedge (P \in U)) \Rightarrow (P \in B)]$

$(\exists P \in C)[((P \in W) \wedge (P \in U)) \wedge (P \notin B)]$

There is an ugly witch who is not bad.

Mathematical Examples

I. The integer n is a perfect square.

Solution: $\exists m \in \mathbb{Z}\ [\ m^2 = n\ ]$

II. The number 2 is not the square of any integer.

Solution: $\neg\{\ \exists m \in \mathbb{Z}\ [\ m^2 = 2\ ]\ \}$
$\Leftrightarrow \forall m \in \mathbb{Z}\ [\ m^2 \neq 2\ ]$

More Mathematical Examples

III. p is a prime number.

Solution: First write "p is a composite number"

$$\exists m \in \mathcal{P}\ \exists n \in \mathcal{P}\ [\ mn = p\ ],$$

where $\mathcal{P} = \{2, 3, 4, \ldots\}$. Negate this sentence:

$$\begin{aligned}
&\neg\{\ \exists m \in \mathcal{P}\ \exists n \in \mathcal{P}\ [\ mn = p\ ]\ \} \\
&\Leftrightarrow \forall m \in \mathcal{P}\ \neg\{\ \exists n \in \mathcal{P}\ [\ mn = p\ ]\ \} \\
&\Leftrightarrow \forall m \in \mathcal{P}\ \forall n \in \mathcal{P}\ [\ mn \neq p\ ]
\end{aligned}$$

Express these sentences symbolically

1. The integer n is odd.
2. There is no largest real number.
3. The quantity $\lim_{x\to\infty} \cos x$ does not exist.

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(IMACS)

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