The Remarkable Life of the Isoperimetric Problem: The World's Most Influential Mathematics Problem

Richard A. Tapia

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Rice University School Math Project
Houston

Outline

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Question

What math problem has had the greatest impact on society and the production of new mathematics?

Candidate: Fermat's Last Theorem

$$a^n + b^n = c^n$$

has no integer solutions for integer $n \geq 3$
No, it has the largest number of incorrect proofs.

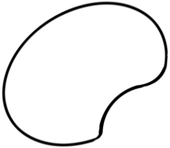
Note: Fermat's Last Theorem is so well-known and respected because its proof is so difficult, not because of its influence or importance in mathematics.

Note: The mathematicians trap: importance of result is directly proportional to difficulty of proof. Consequence — when writing a research paper choose the difficult proof, when writing a book choose the simple proof.

Answer

Isoperimetric Problem:

- Determine from all simple closed planar curves of the same perimeter, the one that encloses the largest area.



- Even the uninitiated correctly conjecture that the solution is a circle.

Note: It also has had a large number of incorrect solutions, in particular, necessity used as sufficiency.

Note: It is clearly the world's best known mathematics problem; and it is unique among math problems in that ancient and medieval poets and historians routinely incorporated it into their writings. Perhaps this is because the problem is so easy to state and understand and has interesting applications.

Remarks associated with the Isoperimetric Problem

- The issue of the relationship between perimeter and area was important to the early Greeks. It was of particular importance for land management and led to clever cheating.
- 400 BC Thucydides (historian and public official), and others, measured the size of a city by the time that it took to circumnavigate the city.
- 400 AD Proclus (mathematical historian)
greatly mocked his Greek forefathers for "measuring the size of a city by the length of its walls, and the size of an island by the time it took to sail around it."
- Ancient Greek philosophers believed that the creator solved the isoperimetric problem in 3-D when He/She created the Earth.

Zenodorus (Greek mathematician who lived 200–120 BC)

1. wrote work entitled *On Isometric Figures*
2. work lost, but referenced by Pappus and Theon 300 years later
3. studied figures with equal perimeters and different shapes
4. proved (stated for sure)
   - the circle has a greater area than the regular polygon of the same perimeter
   - of two regular polygons of the same perimeter, the one with the greater number of angles has the greatest area
5. conjectured (without attempting a proof)
   - in 2-D the circle solves the Isoperimetric Problem
   - in 3-D the sphere solves the Isoperimetric Problem

Queen Dido and the Isoperimetric Problem

Virgil the Roman poet (29–19 BC) in the epic Latin poem *The Aeneid* writes of the Tyrian Princess, Dido whose life was in danger and therefore fled her homeland by ship with her wealth and entourage. They crossed the Mediterranean sea and arrived on the shores of what today is modern Tunisia in North Africa. Dido spied a spot that would be perfect for her and her small group. The natives were not too happy about newcomers, but Dido was able to make a deal with their king, Japon. She promised him a fair amount of money for as much land as she could mark out with a bull's hide. The king thought that he was getting the better end of the deal, but soon realized that the woman he was dealing with was much smarter than he had expected.

The trick that she employed was to have her people cut the hide up into very thin strips which they sewed together into one long string. They then took the seashore as one edge for the piece of land and laid out the string in the form of a semi-circle, giving them a much bigger piece of land than the king had thought possible. They called their community Carthage, which translates in their native Phonician to “New Land” in memory of their Tyrian origin. King Japon was very impressed with Dido’s mathematical intelligence and asked her to marry him. She refused; so he had a university built, hoping to attract a future wife of similar mathematical talent.

REMARK: This is perhaps the first formal documentation of the 1960's "Tune In, Turn On, Drop Out" counterculture icon and Harvard Professor Timothy Leary's[1] adage—"intelligence is an aphrodisiac".

The Aeneid continues telling us that as Queen of Carthage, Dido received Trojan exiles with hospitality. The Trojan prince Aeneas, protagonist of the Aeneid, met her on his way from Troy to Livinium (now Rome). She fell in love with him, and when he spurned her love and left to fulfill his destiny she committed suicide.

REMARK: "mathematical intelligence is neither necessary nor sufficient to deal with the trials and tribulations of the heart".

[1] fired from Harvard for allowing undergraduate students to engage in his experiments on effective use of drugs

Dido Purchases Land for the Foundation of Carthage. Engraving by Matthäus Merian the Elder, in *Historiche Chronica*, Frankfurt a.M., 1630. Dido's people cut the hide of an ox into thin strips and try to enclose a maximal domain.

Application of the Dido Maximum Principle

Medieval map of Cologne

Yet Another Application of the Dido Maximum Principle

Medieval map of Paris

- 1691 — Bitter argument between Johann and Jakob Bernoulli concerning Johann's incorrect solution of the isoperimetric problem.

- 1744 — Euler builds multiplier theory to solve the isoperimetric problem. He realizes that he has only demonstrated that if a solution exists, then it must be the circle, i.e., multiplier theory is necessary, but not sufficient for a solution to exist.

As a consequence of multiplier theory Euler observes that the following two problems are equivalent (they have the same solutions).

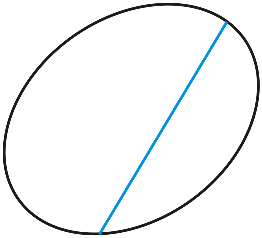
The isoperimetric problem: Determine from all simple closed planar curves that have the same perimeter, the one with the largest area.

The iso-area problem: Determine from all simple closed planar curves that enclose the same area, the one with the smallest perimeter.

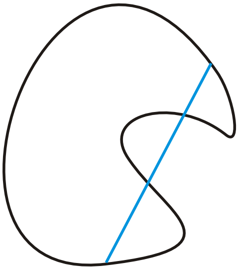
- 1838 — Jacob Steiner constructs a geometric proof for the isoperimetric problem. This proof gained great visibility in the mathematics community and has been called a model of mathematical ingenuity by math historians.

Note: I find it interesting that most presentations of Steiner's proof in the contemporary elementary literature are sloppy and/or flawed.

Convex Set S



Convex



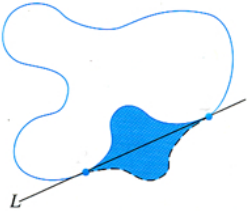
Not Convex

Definition

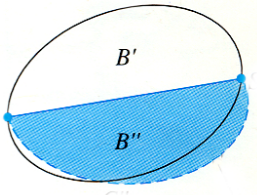
For $x, y \in S \quad \alpha x + (1-\alpha)y \in S \quad \forall \quad \alpha \in [0,1]$.

The Steiner 3 Step Proof

Step 1 The curve must be convex:

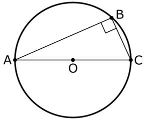


Step 2 Perimeter bisector divides the curve into equal areas and we can therefore symmetrize across the bisector

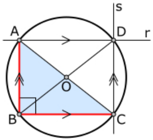


Preliminary Results from Geometry needed to complete Steiner's Proof

Lemma 1 (Thales Theorem 600 BC) If AC is a diameter, then the angle at B is a right angle.



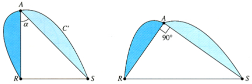
Lemma 2 (Converse to Thales Theorem) A right triangle's hypotenuse is a diameter of its circumcircle.



Lemma 3 Of all possible triangles with two sides of specified lengths the triangle of maximum area is the right triangle.

The Steiner 3 Step Proof continued
"STEINER SYMMETRIZATION"

Step 3   All inscribed angles determined by the perimeter bisector must be right angles.



Proof of Steiner's Theorem:

If angle α is not 90°, then according to Lemma 3 we can increase the area without increasing the perimeter. The proof that the solution of the isoperimetric problem is the circle now follows from the converse to Thales Theorem (Lemma 2), since every point on the curve will determine a radius.

Peter Dirichlet: “Jakob, you have an incomplete proof. You have assumed that a solution exists.”

Jakob Steiner: “Peter, I have a valid proof and I’m not going to let you rain on my parade, please go away.”

Remark: It is remarkable that the many proofs up to this time were either incomplete, incorrect, or implicitly used necessity as sufficiency.

- 1879 — Weierstrass gives first complete proof in his university lectures.

Note: Weierstrass promoted cleanliness of statement, rigor of proof, and sufficiency theory. He was a brilliant mathematician, taught secondary school, and published his first significant paper when he was 40 years old.

Summary: Impact on Contemporary Mathematics

The isoperimetric problem (embedded in controversy) lead to the early calculus of variations. Key players were Fermat, Newton, Johann Bernoulli, and Jakob Bernoulli.

The early calculus of variations led to the golden era of mathematics that we recognize as the 18th and 19th centuries. Essentially all mathematical names that we recognize from that time period attempted work in the calculus of variations.

Optimization: The Cradle of Contemporary Mathematics

Optimization problems are relatively easy to understand when compared with problems in many other branches of mathematics. Controversy invariably leads to interest. Hence, important optimization problems embedded in some controversy have played major roles in motivating and promoting mathematical activity. Mathematical, indeed scientific, activity can be motivated by many factors, and not all are removed from human emotion, as some might have us believe.

PART 2

- Fermat's Basic Optimization Principle
  (The world's first optimization principle)

- Abstracting the Variational Equality
  (The world's second mathematical optimization principle).

Fermat's Basic Optimization Principle (1628)

$$\text{If } x^* \text{ minimizes } f : \mathbb{R} \to \mathbb{R}, \text{ then } f'(x^*) = 0.$$

Note: This principle preceeds the formal definition of the derivative.

Remark: In contrast to Fermat's Last Theorem, we call this Fermat's First Theorem because:

- It is so easy to prove
- It is the most useful and (most used) theorem among all his theorems (conjectures)

Some interesting historical information on Fermat

- Lived 1601–1665
- Born and educated and functioned as a lawyer and legal official in Toulouse, France
- Did mathematics for recreation and considered it a hobby. He dabbled and rarely produced proofs. As such he was sloppy and chose not to include detail or polish his work. Much of his work required “a fill in the blanks” activity as is characterized by his last theorem. He did not publish and communicated his work in the form of letters to important people.
- Directly and naively engaged in controversy with prominent mathematicians of the times and in particular with the powerful Descartes.

- Conceived and applied the differential calculus in a 1628 unpublished work entitled *Minima and Maxima and the tangent to a curve*. Note that this was 15 years before Newton was born and 18 years before Leibniz was born.
- Ten years later in 1638 he made his work semi-public in a letter to Descartes. The work stepped strongly on the toes of work that Descartes was doing on tangents to curves.
- When asked for an official evaluation of Descartes work he wrote "he is groping around, in the shadows."
- Descartes responded with the public statement "Fermat is inadequate as a mathematician and as a thinker." This damaged Fermat's reputation.
- In prominent competition between Descartes and Fermat on the notion of a tangent to a curve, Fermat won and Descartes lost.
- After the dust settles Descartes writes to Fermat "your work on tangents is very good, if you had explained it well from the onset, I would not have had to criticize it or you."

- Fermat had little to no interest in physical applications of mathematics, he just loved the math. This is in strong contrast to Newton who did the mathematics for his love of physical applications. Even his notation and terminology showed this.
- 1642 Herigone in *Cursus Mathematicus* adds a supplement containing Fermat's work on minima and maxima and tangents to curves.
- 1679 Fourteen years after his death Fermat's collected works were published (son played a major role) and included nine papers on minima and maxima.
- E.T. Bell in *Men of Mathematics* calls Fermat "the prince of amateurs."

Supplementary Remark on the development of the calculus

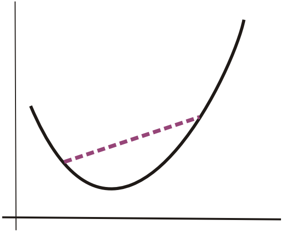
The derivative and the integral were known for many years before Newton or Leibniz are said to have invented the calculus. The primary component in the calculus is what we call today the fundamental theorem of calculus. It was first stated in 1669 by Issac Barrow, a teacher of Newton. Newton and Leibniz built a rather complete understanding of the calculus sometime between the years 1670 and 1700. Formal definition, rigorous statement, and rigorous proof was made by Cauchy in 1823. The following quote from E.T. Bell's *Men of Mathematics* is critical to the development of new mathematics:

> *"If the inventors of the calculus had had to worry about detail and formality of definition and proof it would never have been invented."*

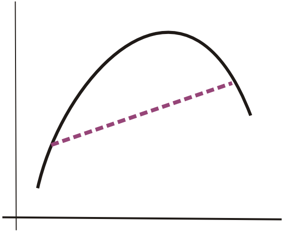
PART 3

The application that never was and could and should have been made by Euler or Lagrange: Identification of convexity and the early and proper solution of the isoperimetric problem.

Convex Function J



Convex



Not Convex

Definition

$$J(\alpha x_1 + (1-\alpha)x_2) \leq \alpha J(x_1) + (1-\alpha)J(x_2) \quad \forall \quad \alpha \in [0,1].$$

An Important Observation

Theorem 1

For a differentiable convex function the derivative equal to zero is both necessary and sufficient for a point to be a minimizer.

Proof

Consider y and y^* in domain of J and $J'(y^*) = 0$. Then,

$$\begin{aligned}
0 = J'(y^*)(y - y^*) &= \lim_{t \downarrow 0} \left[\frac{J(y^* + t(y - y^*)) - J(y^*)}{t} \right] \\
&= \lim_{t \downarrow 0} \left[\frac{J((1 - t)y^* + ty) - J(y^*)}{t} \right] \\
\text{By convexity} \qquad &\leq \lim_{t \downarrow 0} \left[\frac{(1 - t)J(y^*) + tJ(y) - J(y^*)}{t} \right] \\
&= J(y) - J(y^*)
\end{aligned}$$

Therefore,

$$J(y^*) \leq J(y) \quad \text{for all} \quad y.$$

The application that could have been made by Euler or Lagrange.

Solution of the Isoperimetric Problem using convexity
(we work with the semi-circle as Queen Dido did)

Iso-Perimeter Problem

$$\max J(y) = \int_{-a}^{a} y(x)dx$$

subject to

$$y(-a) = y(a) = 0$$
$$\int_{-a}^{a} \sqrt{1 + y'(x)^2}dx = \ell$$
$$y \in C^1[-a, a]$$

Iso-Area Problem

$$\min J(y) = \int_{-a}^{a} \sqrt{1 + y'(x)^2}dx$$

subject to

$$y(-a) = y(a) = 0$$
$$\int_{-a}^{a} y(x)dx = A$$
$$y \in C^1[-a, a].$$

Equivalence Result

Iso-Perimeter **Iso-Area**

$$(IP) \quad \begin{aligned} &\max \ A(y) \\ \text{S.T.} \ &\ell(y) = \ell \\ &B(y) = 0 \end{aligned} \qquad (IA) \quad \begin{aligned} &\min \ \ell(y) \\ \text{S.T.} \ &A(y) = A \\ &B(y) = 0. \end{aligned}$$

Proposition
(IP)$\Leftrightarrow$(IA)
(Have same solutions for compatible choices of ℓ and A)

Proof $\Leftarrow$ (Zhengzheng Feng and Chenbo Li) on 2008 CAAM 560 midterm exam)

Let y_A be a solution of (IA).
If y_A is not a solution of (IP) with $\ell = \ell(y_A)$ then $\exists\ y_p$ s.t.

$$A(y_p) > A(y_A) \quad \text{and} \quad \ell(y_p) = \ell(y_A).$$

Observe that by the monotonicity of the integral $A(\alpha y)$ and $\ell(\alpha y)$ are monotone increasing in α.
Choose $\alpha < 1$ so that $A(\alpha y_p) = A(y_A)$
then $\ell(\alpha y_p) < \ell(y_A)$.
Contradiction since y_A solves (IA).

Proof of $\Rightarrow$ is similar.

Remark: The proofs in the literature work with the circle.

Theorem 2 Consider the Iso-Area Problem. Let y be a feasible function:

$$y(-a) = y(a) = 0 \quad \text{and} \quad \int_{-a}^{a} y(x)dx = A.$$

Let η be an admissible function:

$$\eta(-a) = \eta(a) = 0 \quad \text{and} \quad \int_{-a}^{a} \eta(x)dx = 0.$$

Consider $\phi(t) = J(y + t\eta)$
where $J(y) = \int_{-a}^{a} \sqrt{1 + y'(x)^2} dx$.
Then $\phi : \boldsymbol{R} \to \boldsymbol{R}$ is convex and $\phi(0) = J(y)$.
Moreover, $\phi'(0) = 0$, for all admissible η, is both a necessary and sufficient condition for y to solve the iso-area problem.

Proof: The proof is the same as the proof of Theorem 1.

We now use our Theorem 2 to show that the semi-circle

$$y(x) = \sqrt{a^2 - x^2} \quad \text{for} \quad -a \leq x \leq a$$

solves the Iso-Area problem for the choice $A = \frac{\pi}{2}a^2$ and therefore solves the Iso-Perimetric problem for the choice $\ell = \pi a$.

For an admissible variation, i.e.,

$$\eta(a) = \eta(-a) = 0, \quad \int_{-a}^{a} \eta(x)dx = 0.$$

Consider $\phi(t) = J(y + t\eta)$
where

$$J(y) = \int_{-a}^{a} \sqrt{1 + y'(x)^2}dx.$$

Calculation gives

$$\begin{aligned}
\phi'(0) &= \int_{-a}^{a} \frac{y'(x)\eta'(x)}{\sqrt{1 + y'(x)^2}} \\
&= -\int_{-a}^{a} x\eta'(x)dx \\
&= -x\eta(x) \Big|_{-a}^{a} + 1/a \int_{-a}^{a} \eta(x)dx \\
&= 0
\end{aligned}$$

Hence the semi-circle solves the iso-area problem and in turn the equivalent iso-perimetric problem.

Remark: We have just solved the world famous isoperimetric problem using elementary tools known to Euler and certainly Lagrange. Of course we had to realize that convexity gave sufficiency.

A change of pace:

Is this an application of the iso-perimeter or the iso-area principle?

Medieval map of Cologne

Was area or length the design parameter?

Medieval City Planner:

We can afford to build a wall of length ℓ, so let's build it in the form of a semi-circle and maximize area

or

We need a city of area A, so lets build the wall in the form of a semi-circle to minimize cost

Vote for one.

PART 4

SUMMARY

Dear Professors Euler and Lagrange:
You are two of the greatest mathematicians of all time and certainly the two greatest mathematicians of the eighteen century. Therefore my comments below are made with great respect and even greater humility. I do hope that you will find them informative and perhaps even amusing. I will restrict my comments to the early calculus of variations. To start with, I compliment both of you for seeing the need for theory for a general class of problems, you seem to be among the first in the calculus of variations. In this work, Leonhard, you were bold and creative, but rather sloppy in proof. I find it interesting that throughout your life you cautioned others about being cautious when working with infinite processes, but failed to observe it in much of your own work. However, these flaws do not really bother me, and I will tell you why. In these instances you had deficient proofs of correct results. This is much better than having a flawed proof of an incorrect result. It is important that you were always correct in your statements.

Joseph, in addition to all your great original work you functioned, for actually much of your life, as the closer, the polisher, the one who made everything clean, clear , and correct. In 1777 you said in a letter to Laplace "I enjoy the works of others much more than my own, with which I am always dissatisfied". You served in this capacity with the calculus. You were the conduit between Newton and Leibniz, the originators, and Cauchy the final polisher of the calculus.

Now let's turn to the legendary isoperimetric problem. Leonhard, I am told that you memorized Virgil's Aeneid from cover to cover and that you could tell people the first and last word on every page. This is hard to believe, but even if partially true it shows that Queen Dido's isoperimetric problem was deeply embedded in your subconscious. So you built multiplier theory to solve this problem and realized that you only had necessity, i.e. if the problem had a solution it had to be the circle. This is good because so many mathematicians of the time failed to make real distinctions between necessity and sufficiency. However, the irony is that in this case you really did have sufficiency. All you had to do was be aware of convexity and that in its presence necessity becomes sufficiency in both the variational equality and the multiplier theory that you built.

However, this observation only applies to the iso-area problem. But that would not be a problem since you are the one that pointed out that multiplier theory for the isoperimetric problem and the iso-area problem is the same — if the circle solves one it solves the other. Here Joseph I have to look at you. You were the closer, you had all the understanding plus the demonstrated technique for doing exactly this. Was convexity not known to you? A simple picture would have lead to the conjecture. Of course you could both say why do I not look at Fermat, after all he was the one who spent all the time on minimization in one dimension, and there for sure a picture should have led to the result. I thought of that, and I think that Fermat, the loose cannon that he was, may have thought of it and failed to write it down or write it up. So Leonhard and Joseph you were just an eyelash, a blink of an eye, an epsilon away from giving the world its first valid proof that the circle solves the isoperimetric problem. But, no we had to wait a hundred and fifty years to see the first proof. I would have much preferred to see a proof that used the very tools that you both embraced and developed.

Indeed Joseph it made me very happy to read that, of all your great work, you considered your early work in the calculus of variations, thought out when you were nineteen, your masterpiece. You said that it was by means of this calculus that you unified mechanics. This unification prompted Hamilton to say "you made of mechanics a scientific poem".

Thank you for giving the world so much and giving me such interesting material to think about and to talk about.

Regards,

Richard Tapia
University Professor