RUSMP/MLI Colloquium

Tropical Mathematics

An Interesting and Useful Variant of Ordinary Arithmetic

June 8, 2005

Tropical Mathematics

A new mathematics

- Starts with a new arithmetic
- Includes polynomials, curves, higher algebra
- Useful in combinatorics, algebraic geometry
- Useful in genetics
- It is fun to do math in a different setting

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Why Tropical Mathematics?

- Coined by French mathematicians
- In honor of Imre Simon, a Brazilian mathematician
- The name simply reflects how a few Frenchmen view Brazil

Tropical Arithmetic

- Ordinary arithmetic
  - Real numbers, addition $(+)$ and multiplication $(\times)$
- Tropical arithmetic
  - Real numbers plus infinity, denoted by ∞
  - Tropical addition $(\oplus)$
  - Tropical multiplication $(\otimes)$

Tropical Addition

$a \oplus b =$ the minimum of a and b.

- Examples:

$$3 \oplus 5 = 3, \quad 3 \oplus (-5) = -5$$
$$12 \oplus 0 = 0, \quad 0 \oplus (-3) = -3$$

- The additive unit is ∞.
  - $\infty \oplus 3 = 3$
  - $\infty \oplus x = x \oplus \infty = x$ for all x

Tropical Addition Table

$\oplus$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2
3	1	2	3	3	3	3	3
4	1	2	3	4	4	4	4
5	1	2	3	4	5	5	5
6	1	2	3	4	5	6	6
7	1	2	3	4	5	6	7

Differences

- Subtraction is not always possible.
  - The equation $3 \oplus x = 5$ has no solution.
  - The equation $3 \oplus x = 1$ has a solution.
  - The equation $a \oplus x = \infty$ has no solution if $a \neq \infty$.
- We have to stay away from looking for solutions to equations.

Tropical Multiplication

- $a \otimes b = a + b$
  - Tropical multiplication is the same as ordinary addition.
- Examples:

$$3 \otimes 5 = 8, \quad 3 \otimes (-5) = -2,$$
$$(-1) \otimes 3 = 2, \quad 1 \otimes 13 = 14.$$

- The multiplicative unit is 0.
  - $0 \otimes 13 = 13.$
  - $0 \otimes x = x \otimes 0 = x$ for all x.

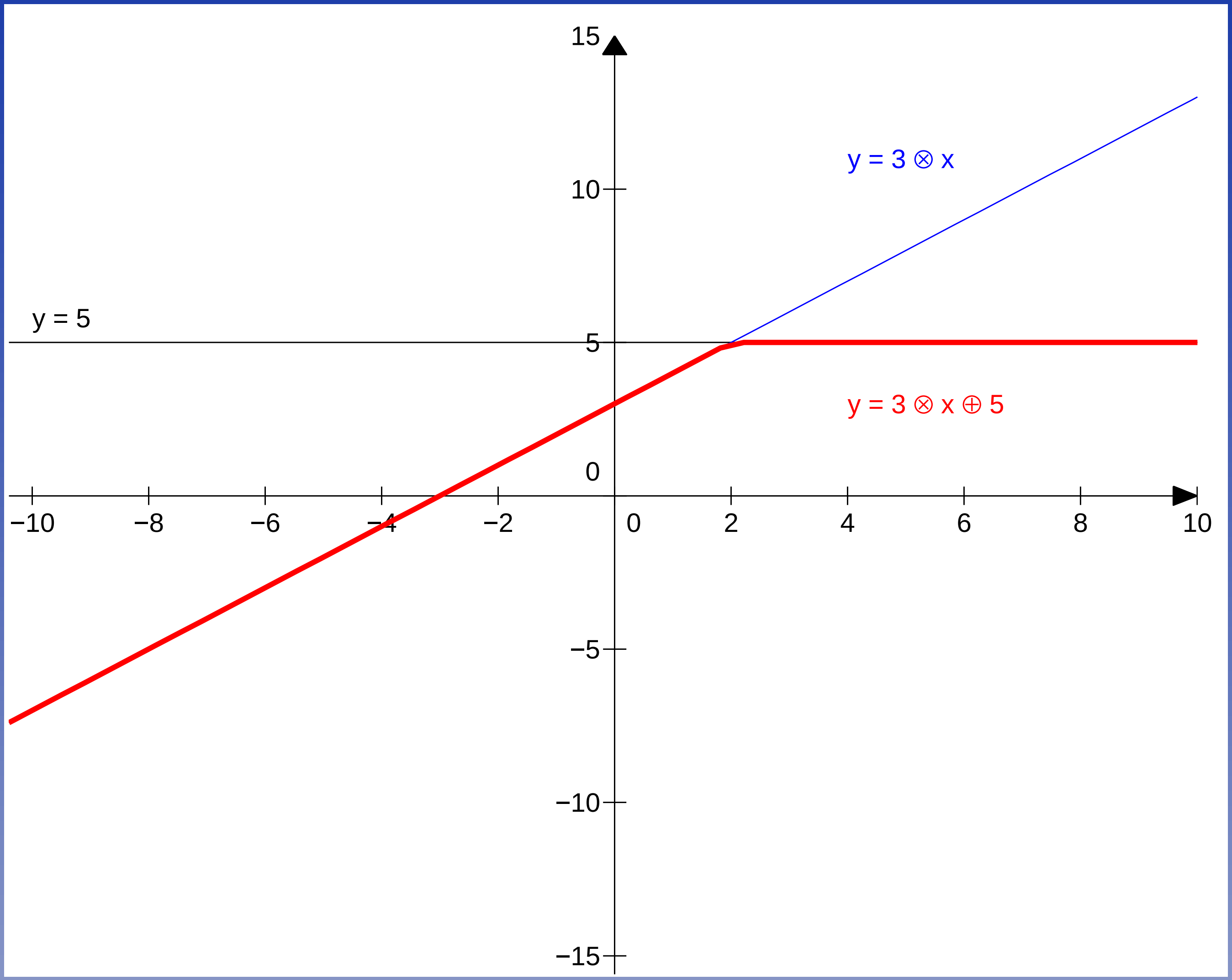
Tropical Multiplication Table

$\otimes$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	7
2	2	3	4	5	6	7	8
3	3	4	5	6	7	8	9
4	4	5	6	7	8	9	10
5	5	6	7	8	9	10	11
6	6	7	8	9	10	11	12

Similarities and Differences

- Commutative laws are valid
- The distributive law still holds.
- $(x \oplus y)^3 = x^3 \oplus y^3$

Linear Functions



Linear Functions

- The graph of $y = 5$ is a straight line with slope 0.
- The graph of $y = 3 \otimes x$ is a straight line with slope 1.
- The graph of $y = 3 \otimes x \oplus 5$ is a crooked line.
- Notice:

$$\begin{aligned} 3 \otimes x \oplus 5 &= \min\{x + 3, 5\} \\ &= 3 + \min\{x, 2\} \\ &= 3 \otimes (x \oplus 2) \end{aligned}$$

  - $x = 2$ is where the graph bends.

Monomials

- Monomials:

$$x^2 = x \otimes x = x + x = 2x$$

$$x^3 = x \otimes x \otimes x = 3x$$

$$x^p = p \times x$$

  - Monomials are linear functions with integer coefficients.
- $3 \otimes x^2 = 3 + (2x)$
  - The graph is a line with slope 2.
- $4 \otimes x^3 = 3x + 4$
  - The graph is a line with slope 3.
- The exponent is the slope of the graph.

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Polynomials

Example 1:

$$p(x) = 2 \otimes x^2 \oplus x \oplus 5$$
$$= \min\{2x + 2, x, 5\}$$

- The graph is a twice bent line.
  - The graph bends at $x = -2$ and $x = 5$.
- We can show that $p(x) = 2 \otimes [x \oplus (-2)] \otimes [x \oplus 5]$

Polynomials

Example 2:

$$p(x) = x^2 \oplus 3 \otimes x \oplus 2$$
$$= \min\{2x, x+3, 2\}$$

- The graph is a once bent line.
  - The graph bends at $x = 1$
- We can show that $p(x) = (x \oplus 1)^2$

Factorization of Polynomials

- Our two example polynomials factor into linear factors.
  - The factors have the form $x \oplus a$, where a is a bend point for the graph.
- Any tropical polynomial can be expressed in one and only one way as the product of linear factors.
  - Thus the Fundamental Theorem of Algebra remains true in tropical mathematics.
  - The factors are of the form $x \oplus a$, where a is a bend point for the graphs of the function. All such factors occur.

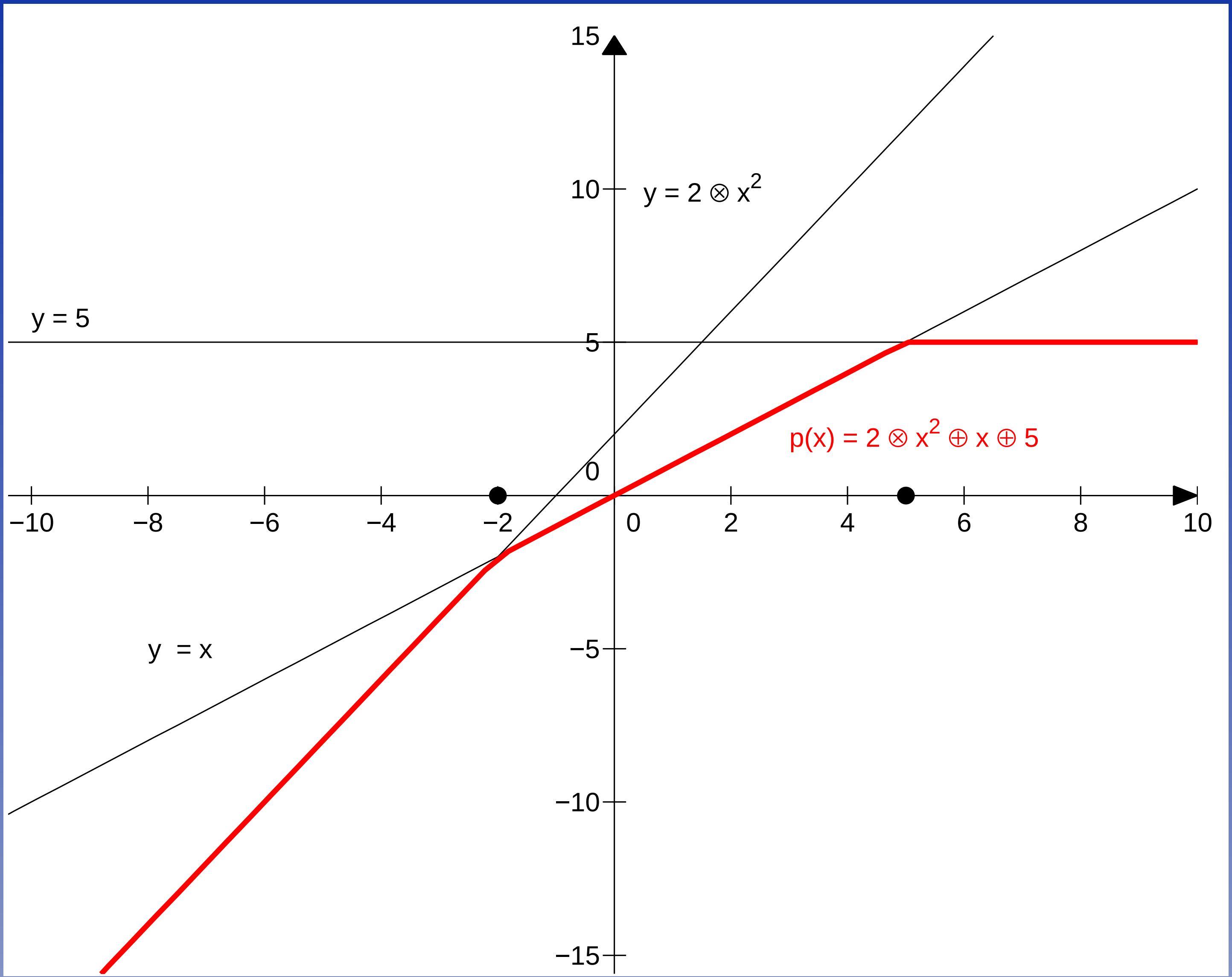
Polynomials in Two Variables

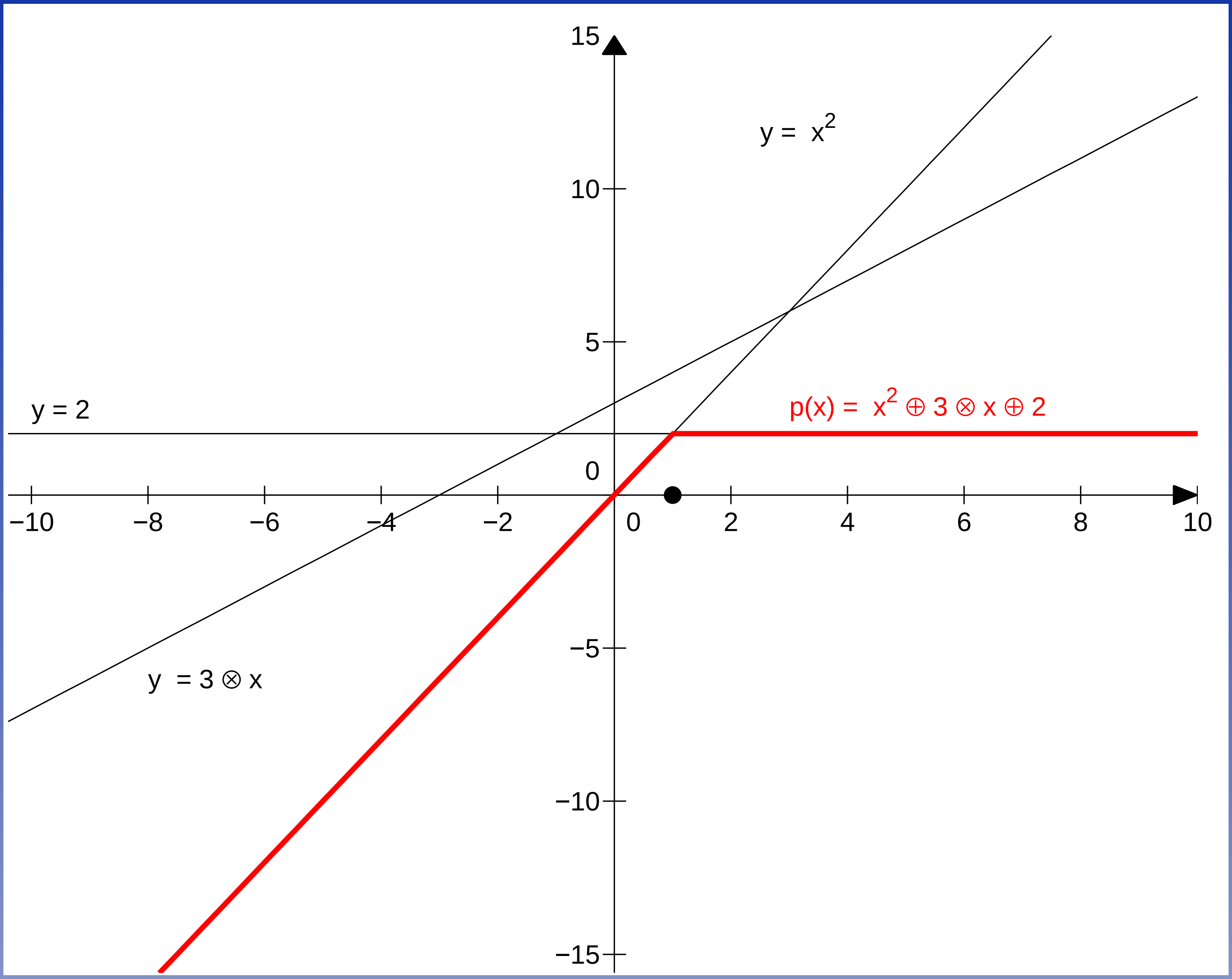
- A monomial represents a linear function.
  - Example: $p(x, y) = 3 \otimes x \otimes y = 3 + x + y$
- A polynomial represents the minimum of one or more linear functions.
  - Example: $p(x, y) = x \oplus y \oplus 1 = \min\{x, y, 1\}$
- The bend points of the graph occur where two or more of the linear functions agree.

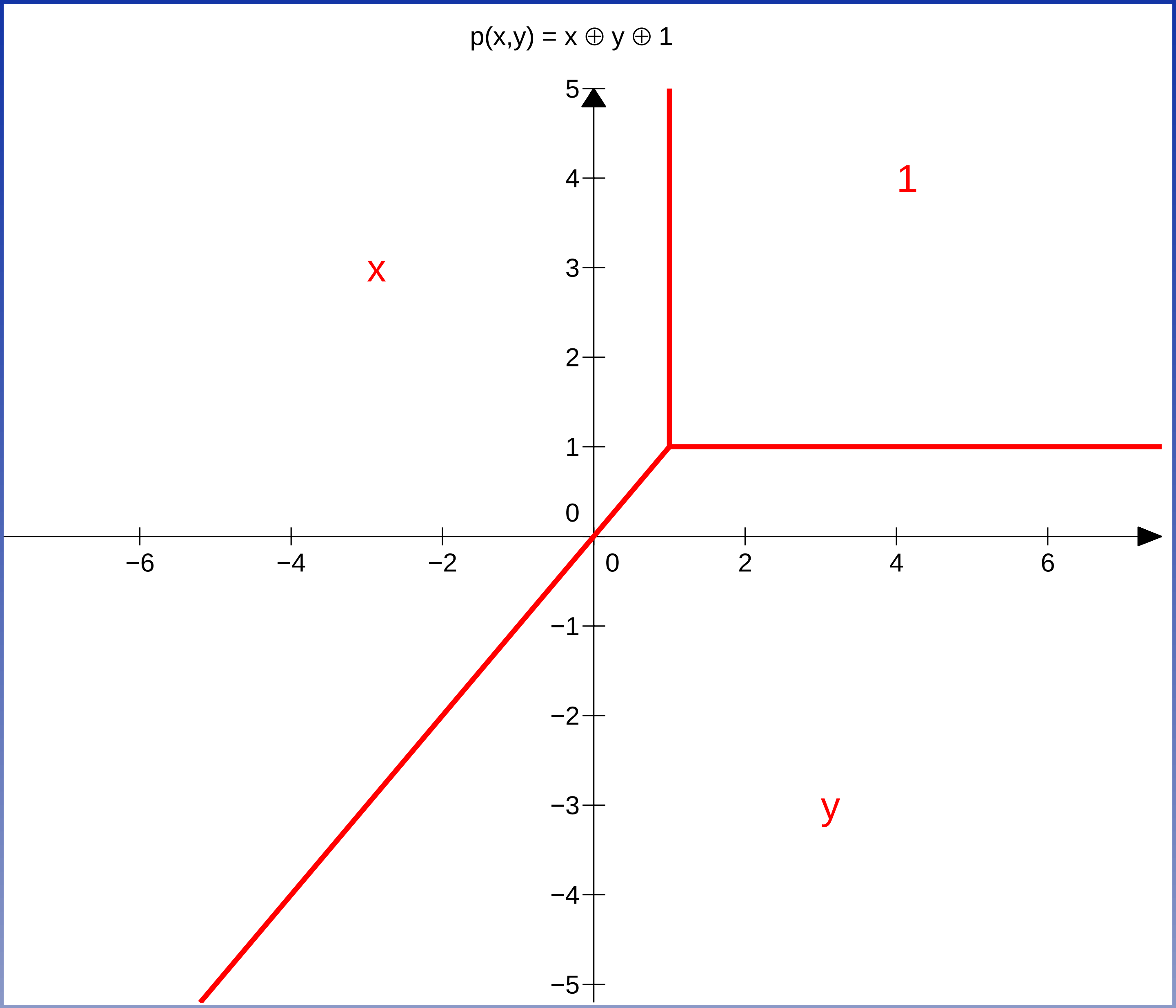
Curves

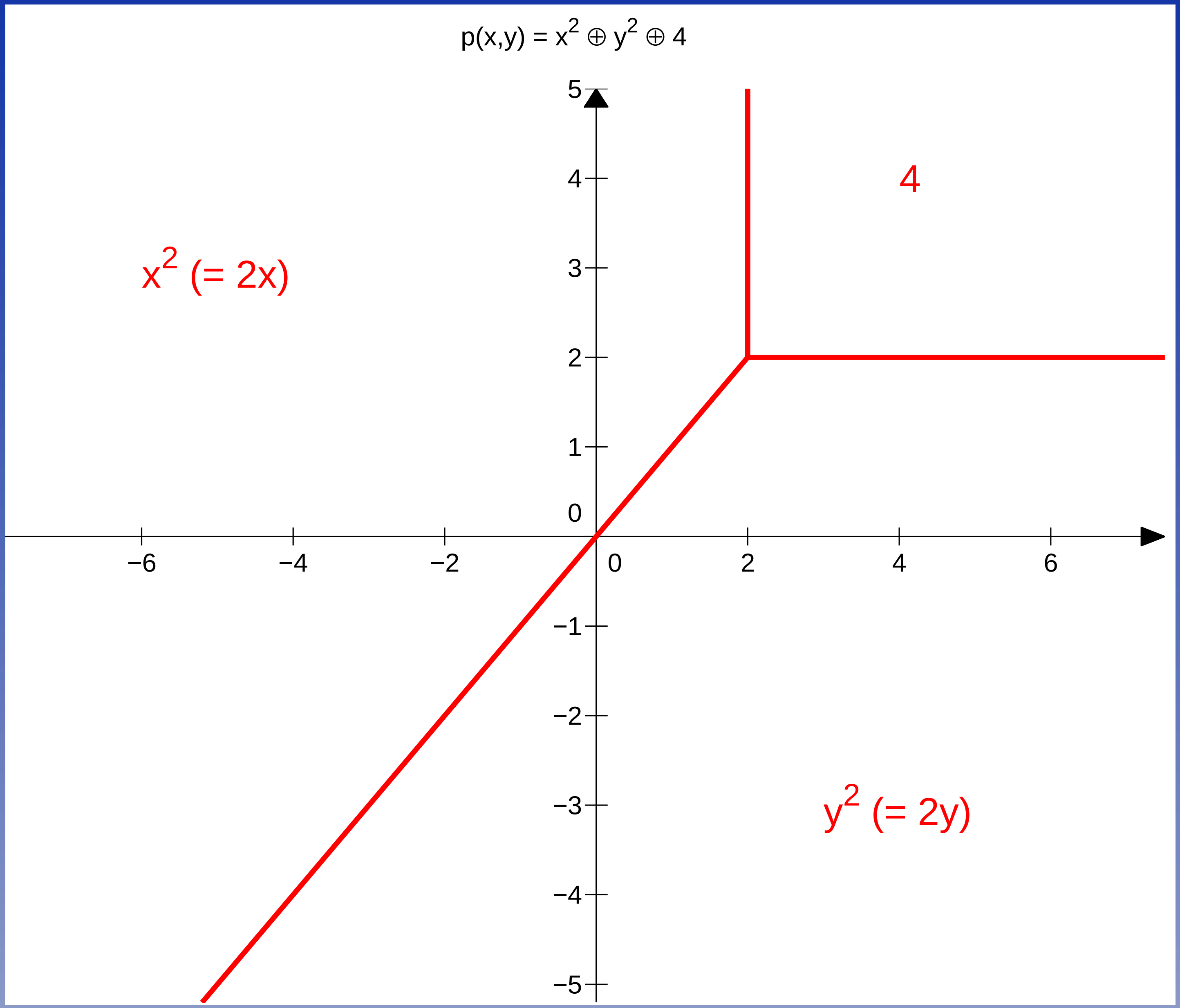
- In ordinary math, the zero set of $x^2 + y^2 - 1$ is a circle — a curve.
- In tropical math, the zero set is replaced with the bend set — a tropical curve.
- Examples
  - 1. $p(x, y) = x \oplus y \oplus 1 = \min\{x, y, 1\}$
  - 2. $p(x, y) = x^2 \oplus y^2 \oplus 4 = \min\{2x, 2y, 4\}$
  - 3. $p(x, y) = x^2 \oplus y^2 \oplus x \oplus 4 = \min\{2x, 2y, x, 4\}$

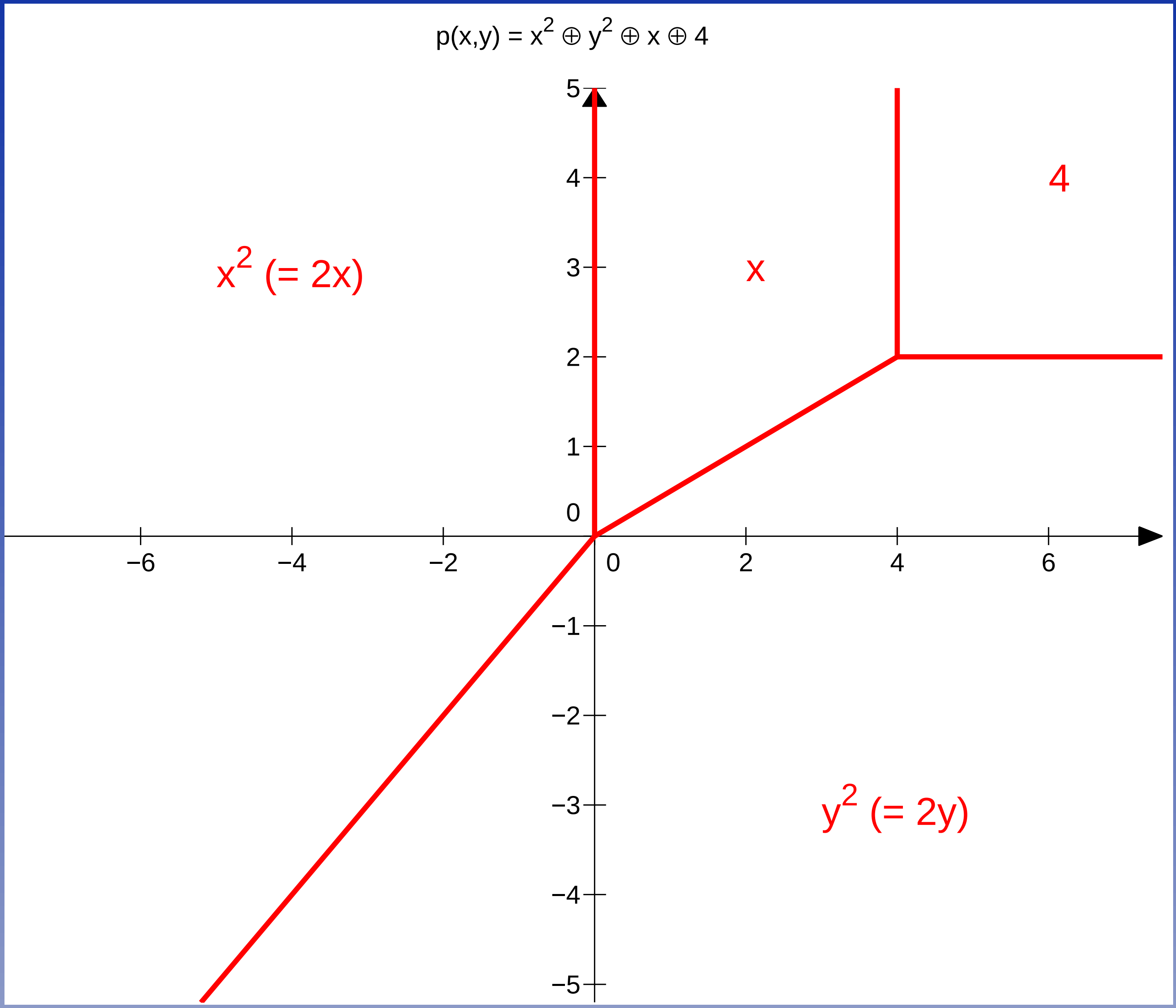
The End


15
10
5
0
−5
−10
−15
−10
−8
−6
−4
−2
0
2
4
6
8
10
y = 2 ⊗ x2
y = 5
p(x) = 2 ⊗ x2 ⊕ x ⊕ 5
y = x


15
10
5
0
−5
−10
−15
−10
−8
−6
−4
−2
0
2
4
6
8
10
y = x^2
y = 2
p(x) = x^2 ⊕ 3 ⊗ x ⊕ 2
y = 3 ⊗ x


p(x,y) = x ⊕ y ⊕ 1
1
x
y
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
−4
−2
0
2
4
6


p(x,y) = x2 ⊕ y2 ⊕ 4
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
−4
−2
0
2
4
6
4
x2 (= 2x)
y2 (= 2y)


p(x,y) = x2 ⊕ y2 ⊕ x ⊕ 4
x2 (= 2x)
x
4
y2 (= 2y)
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
−4
−2
0
2
4
6