Exploring Conics with Graphing Technology
Richard Parr
Rice University School Mathematics Project
rparr@rice.edu

There are several ways to explore conic sections on the TI-84+ graphing calculator. In this presentation we will look at not only the CONICS Application on the TI-84+ calculator but also how to use function, parametric and polar graphing modes to draw more mathematics out of the study of conic section.

First let lets look at conic sections using function mode. For conic sections expressed in standard form, solving the equation for *y* in terms of *x* allows. All conic sections except for parabolas with vertical axes of symmetry can be represented by two functions. For example to graph the ellipse:

$$\frac{(x-3)^2}{16} + \frac{(y+2)^2}{9} = 1$$

We can solve this equation for y resulting in:

$$y = -2 \pm \sqrt{9 - \frac{9(x-3)^2}{16}}$$

Graphing these equations in a standard window results in the following graph:
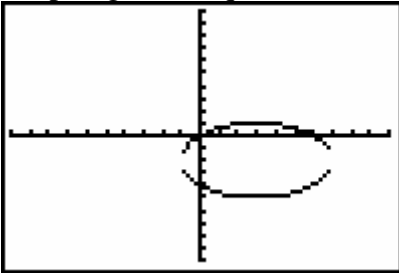
A couple of comments need to be made about this graph in the standard window. First, it is obvious that ellipse does not look like a closed figure. Second it is not in the correct aspect since the units on the *x*-axis are longer than units on the *y*-axis. A much better graph can be viewed by using the viewing window [-9.4, 9.4] x [-6.2, 6.2]:
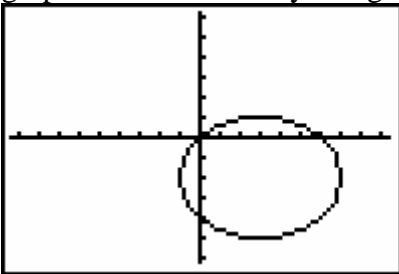
In Pre-Calculus, we can study conic sections that have axes that are not parallel and perpendicular to the coordinate axes. All conic sections have an equation in general form given by:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

This equation can be re-expresses as a quadratic in y:

$$Cy^2 + (Bx + E)y + Ax^2 + Dx + F = 0$$

Using the quadratic formula this can be solved for y:

$$y = \frac{-(Bx+E) \pm \sqrt{(Bx+E)^2 - 4C(Ax^2 + Dx + F)}}{2C}$$

This is a cumbersome expression to enter in the calculator. A nice calculator program can be written that will automatically graph the two functions. The program can be modified to take care of special cases and to give the angle of rotation.

For example graphing the conic section:
$$x^2 + 6xy + y^2 + 8x - 8y - 20 = 0$$

Results in the following graph:
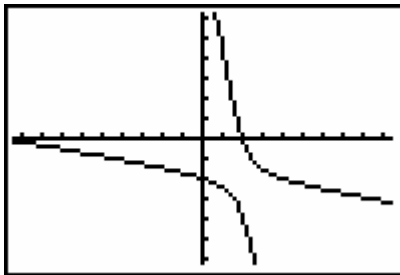
For rotated conic sections, the discriminant $B^2 - 4AC$ determines what type of conic section is present. If $B^2 - 4AC > 0$ the resulting figure is an hyperbola, if $B^2 - 4AC < 0$ the conic is an ellipse and if $B^2 - 4AC = 0$ then the figure is a parabola.

Now let us turn our attention to conics represented in parametric form.
The Pythagorean trigonometric identities allow for easy parametric representation of ellipses and hyperbolas. Parabolas are most easily represented without the use of trigonometry.

Ellipses

A comparison of the Pythagorean identity: $\cos^2 t + \sin^2 t = 1$ and a standard form for the equation of an ellipse : $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ allows for two simple substitutions :

$$\cos^2 t = \dfrac{(x-h)^2}{a^2} \quad \text{and} \quad \sin^2 t = \dfrac{(y-k)^2}{b^2}$$

Solving these two equations for x and y yields a pair of parametric equations:

$$x = a\cos t + h$$

$$y = b\sin t + k$$

A specific example; to graph $\dfrac{(x-3)^2}{9} + \dfrac{(y+2)^2}{4} = 1$ on the TI-83, one would put the calculator in parametric mode. The choice of degree or radian mode is one of personal choice. For the first examples I would do in my class I would use degree mode. Then with a window of [0, 360]$_5$ x [-15, 15] $_1$ x [-10,10] $_1$, the appropriate equations and graphs are shown:
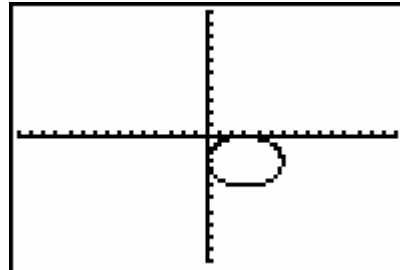
 A few personal comments are important at this point:

I chose substitutions I did to reinforce the use of x and y coordinates of a unit circle to represent sine and cosine respectively. In using this method I am de-emphasizing the idea that "a" corresponds to the major axis, etc. I focus on the idea that "a" is a stretch in the x equation and therefore a horizontal stretch. Likewise, "b" is a vertical stretch. I'd just as soon not use the letters "a" and "b" at all, but focus on the major axis being the axis with the "largest" stretch.

Some students see a contradiction in the transformation in parametric representation when compared to the Cartesian representation, by re-writing $x = 3\cos t + 3$ in the form $x - 3 = 3\cos t$, I try to show that there is really no contradiction.

Hyperbolas

By using the Pythagorean identity: $\sec^2 t - \tan^2 t = 1$ and a standard form for a hyperbola

: $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$

One can derive the following pairs of parametric equations to represent hyperbolas:

$x = a \sec t + h$ $x = b \tan t + h$
$y = b \tan t + k$ $y = a \sec t + k$
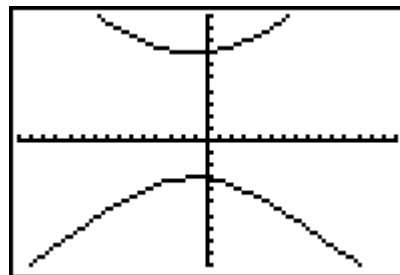(horizontal transverse axis)   (vertical transverse axis)

In a hyperbola, unlike an ellipse, it makes a difference which trigonometric function corresponds with which variable. Therefore using the sane window settings as before to graph the hyperbola $\dfrac{(y-2)^2}{25} - \dfrac{(x+1)^2}{36} = 1$ on the TI-83 we get as a result:

Parabolas

Parabolas are most easily graphed parametrically without the use of trigonometric functions. All non-rotated parabolas can either be written in the form $y = f(x)$ or $x = f(y)$. Parametrically, parabolas that can be written $y = f(x)$ can be graphed using $x = t$ and $y = f(t)$, likewise parabolas that can be represented as $x = f(y)$ can be graphed parametrically using $x = f(t)$ and $y = t$. In this case the T step of the window must be adjusted to include negative values for t or the entire parabola will not appear.

Extensions

Some possible ideas for extension include trying to develop a trigonometric method of graphing parabolas parametrically, and extending the use of parametric graphing to graph rotated conic equations.

Polar graphing of conic sections:
In polar mode, conic sections with one focus at the pole (origin) can be graphed. The general equation of a polar conic can be expressed by the equation:

$$r = \frac{ep}{1 \pm e \sin \theta} \quad \text{(horizontal directrix)}$$

$$\text{or} \quad r = \frac{ep}{1 \pm e \cos \theta} \quad \text{(vertical directrix)}$$

Where e is the eccentricity of the conic and p is the distance from the focus to the directrix. Polar equations of conics can be used to emphasize the definition that eccentricity is the ratio of the distance from a point on the graph to the focus to the distance of the same point to the directrix. If the eccentricity is greater than 1, the graph is a hyperbola, if the eccentricity is between zero and one the graph is an ellipse, and if the eccentricity is 1 the graph is a parabola.

By drawing the directrix in polar form you can illustrate this relationship of the polar conic.

For example:
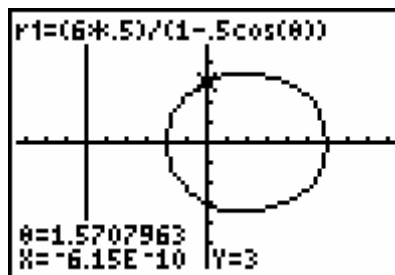
Finally let's look at the conics application. The conics application on the TI-84+ allows us to easily graph conic sections in any of the three ways that we have looked at previously. We will take a tour of the screens that appear when choosing the various modes on the application:

Function mode:

ELLIPSE

1: $\dfrac{(X-H)^2}{A^2} + \dfrac{(Y-K)^2}{B^2} = 1$

2: $\dfrac{(X-H)^2}{B^2} + \dfrac{(Y-K)^2}{A^2} = 1$

ESC

Parametric mode:

ELLIPSE

1: X=Acos(T)+H
   Y=Bsin(T)+K

2: X=Bcos(T)+H
   Y=Asin(T)+K

ESC

Polar Mode:

ELLIPSE

1: $R = \dfrac{2ep}{1-e\cos(\theta)}$

2: $R = \dfrac{2ep}{1+e\cos(\theta)}$

ESC

The conic application does allow you to easily graph conic sections with ease; however the problems with the applications need to be noted:

- Conic sections with axes that are not parallel or perpendicular to the coordinate axes can not be graphed.
- Automatic window can cause make all ellipses look similar
- Difficult to add additional graphs to the graph of the conic sections

Overall there are many ways to study conic sections using the TI-84+ , that can make the once difficult study of 2nd degree relations easier for students to visualize.