Using Parametric Equations to Model Motion

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Cedric French-Galan
Westside High School
nfrench@houstonisd.org

SHIPS IN THE FOG

Situation:

Two ships are sailing in the fog and are being monitored at ten-second intervals by Ensign Pulver. As they enter his radar screen at 6:00 A.M., the Minnow is 1000 meters above the lower left corner of the screen, which will henceforth be referred to as the "Origin". The good ship Lollipop is 8000 meters to the right of the origin.

After ten seconds, the Minnow is 40 meters east and 10 meters north of its original position. The Lollipop is 30 meters west and 20 meters north of its original position.

You may assume that the two ships maintain their courses and speeds throughout the problem.

Problem:

    A. Determine whether the two ships collide.

    1. If they do collide, determine the time and the position of the collision with respect to the origin.

    2. If they do not collide, determine the minimum distance between the ships and when they are closest.

    B. Turn in the work which confirms your conclusion. This may either be an analytical explanation or a description of how technology was used to solve the problem.

    C. Prepare a graph to illustrate the situation. The graph should be labeled in such a way as to be understandable to a casual observer. The names of your group members should be proudly displayed on your graph.

-from Jim Stones

Unit 10 – Parametric and Polar Equations - Classwork

Until now, we have been representing graphs by single equations involving variables x and y. We will now study problems with which 3 variables are used to represent curves. Consider the path followed by an object that is propelled into the air at an angle of 45º. If the initial velocity of the object is 48 feet per second, the object follows the parabolic path given by

$$y = -\frac{x^2}{72} + x$$
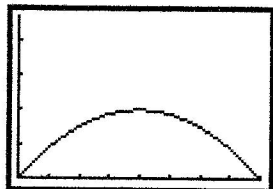
However, although you have the path of the object, you do not know *when* the object is at a given time. In order to do this, we introduce a third variable t, called a **parameter**. By writing both x and y as a function of t, you obtain the parametric equations:

$$x = 24t\sqrt{2} \quad \text{and} \quad y = -16t^2 + 24t\sqrt{2}$$

From this set of equations, we can determine that at the time $t = 0$, the object is at the point $(0, 0)$. Similarly at the time $t = 1$, the object is at the point $\left(24\sqrt{2}, 24\sqrt{2} - 15\right)$.

Definition of a Plane Curve

If f and g are continuous functions of t on an interval I, then the equations
$$x = f(t) \quad \text{and} \quad y = g(t)$$

are called **parametric equations** and t is called the parameter. The set of points (x, y) obtained as t varies over the interval I is called the graph of the parametric equations. Taken together, the parametric equations and the graph are called a plane curve.

When sketching a curve by hand represented by parametric equations, you use increasing values of t. Thus the curve will be traced out in a specific direction. This is called the **orientation** of the curve. You use arrows to show the orientation.

Example 1) Sketch the curve described by the parametric equations:

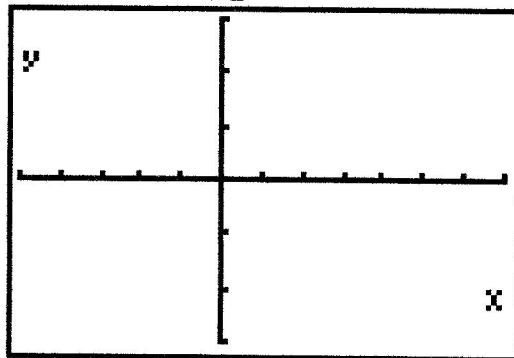$$x = t^2 - 4 \quad \text{and} \quad y = \frac{t}{2} \quad -2 \le t \le 3$$

t	-2	-1	0	1	2	3
x						
y						
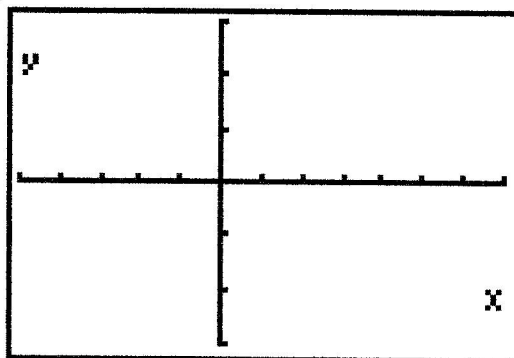
Example 2) Sketch the curve described by the parametric equations:

$$x = 4t^2 - 4 \quad \text{and} \quad y = t \quad -1 \le t \le \tfrac{3}{2}$$

t	-1	-.5	0	.5	1	1.5
x						
y						

Note that both examples trace out the exact same graph. But the speed is different. Example 2's graph is traced out more rapidly. Thus in applications, different parametric equations can be used to represent various speeds at which objects travel along paths.

Note that the TI-84 calculator can graph in parametric mode. Go to MODE and switch to Parametric mode as shown below. Your Y= button now gives you the screen below. The X,T,θ,n button now gives a T when pressed. The equation in example 1 can now be generated. You set the T values by going to your WINDOW and placing them in Tmin and Tmax.

Tstep controls the accuracy and speed of your graph. Large values of Tstep give speed but not little accuracy. Small values of Tstep give a lot of accuracy at the cost of speed. Xmin, xmax, Ymin, Ymax work as before. Note that the arrow showing orientation does not display when you graph a parametric on the calculator. If you are asked to draw a parametric on an exam, you must include it.
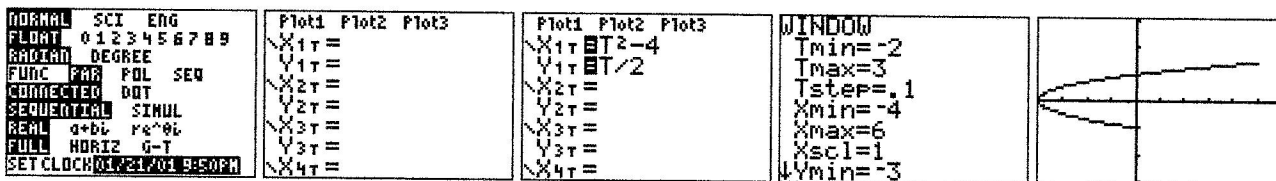
Finding a rectangular equation that represents the graph of a set of parametric equations is called **eliminating the parameter**. Here is a simple example of eliminating the parameter.

Example 3) Eliminate the parameter in $x = t^2 - 4$ and $y = \dfrac{t}{2}$

   • Solve for t in the second equations   • Substitute in the second equations and simplify

Note that both equations give the same graph although they are not plotted in the same direction.
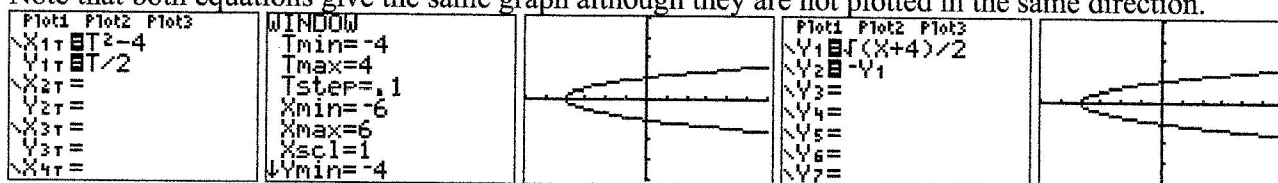
Example 4) Eliminate the parameter in the following parametrics. In each problem, it will usually be easier to solve for t in one equation than the other. Then graph, showing that the 2 equations graph the same curve.

a) $x = t - 3$ and $y = t^2 + \sqrt{t} - 2$

b) $x = 3t + 2$ and $y = \dfrac{1}{2t - 1}$

c) $x = \dfrac{t}{2}$ and $y = \sin(t + 1) - 1$

d) $x = \dfrac{1}{\sqrt{t + 1}}$ and $y = \dfrac{t}{t + 1}$ $\quad t > -1$

Example 5) Sketch the curve represented by $\quad x = 3\cos\theta \quad$ and $\quad y = 3\sin\theta \quad 0 \le \theta < 2\pi$

• Solve for $\cos\theta$ and $\sin\theta$ in both equations.

• Use the fact that $\sin^2\theta + \cos^2\theta = 1$ to form an equation using only x and y.

• This is a graph of an circle centered at (0, 0) with diameter endpoints at (3, 0), (-3, 0), (0, 3), and (0, -3). Note that the circle is traced *counterclockwise* as θ goes from 0 to 2π.

Example 6) Finding parametric equations for a given function is easier. Simply let $t = x$ and then replace your y with t. Find parametric equations for

a) $y = x^2 - 2x + 3$

b) $y = \sqrt{\dfrac{2x - 4}{3x - 1}}$

Note that there are many ways of finding parametric equations for a given function. For the problems above, let $x = t + 2$ and find the resulting parametric equations.
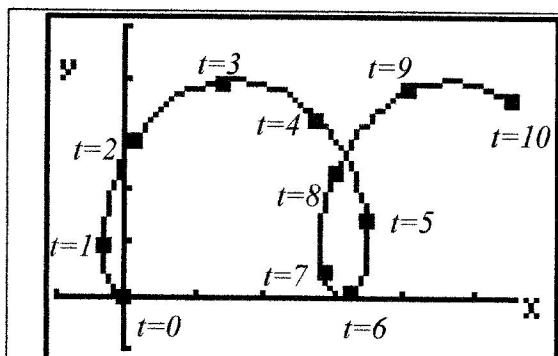
a.

b.

Example 7) At any time t with $0 \le t \le 10$, the coordinates of P are given by the parametric equations:
$$x = t - 2\sin t \quad \text{and} \quad y = 2 - 2\cos t$$
Sketch this using your calculator.

- Points corresponding to integer values of t are shown. At t = 1, P has coordinates of (-.68,. 92); at this instant, P is heading almost due north.
- The full curve is not the graph of a function; some x values have more than one y value.
- The picture shows the x- and y- axes but no t-axis.
- The bullets on the graph appear at equal time intervals but not at equal distances from each other because P speeds up and slows down as it moves.

Example 8) Parametric curves may have loops, cusps, vertical tangents and other peculiar features. Parametric curves are not necessarily functions. Trying to graph the curves below in function mode would necessitate very complex piecewise functions.

Graph a) $x = 2\cos t + 2\cos(4t)$ and $y = \sin t + \sin(4t)$ $\quad 0 \le t \le 5$

b) $x = \sin(5t)$ and $y = \sin(6t)$ $\quad 0 \le t \le 2\pi$

a.

b.
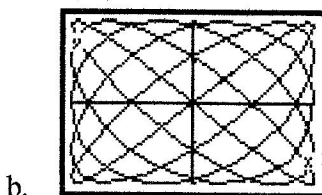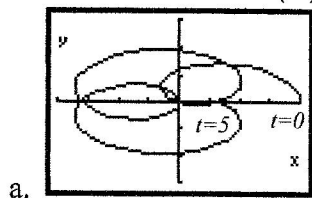
This is called a **Lissajou curve**.

Projectile Motion

If a projectile is launched at a height of h feet above the ground at an angle of θ (measured in degrees or radians) with the horizontal. If the initial velocity is v_0 feet per second, the path of the projectile is modeled by the parametric equations:

$$x = v_0 t \cos\theta \qquad y = h + v_0 t \sin\theta - 16t^2$$

Example 9) For each problem, use the calculator to graph two parametric equations for a projectile fired at the given angle at the given initial speed at ground level. Then use the calculator's trace ability to estimate the maximum height of the object as well as its range.

a.
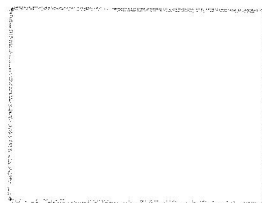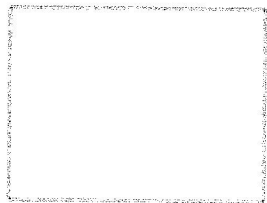$\theta = 30°, v_0 = 90$ ft/sec
$\theta = 30°, v_0 = 120$ ft/sec

b.
$\theta = 30°, v_0 = 90$ ft/sec
$\theta = 70°, v_0 = 90$ ft/sec

Par. equation 1 _____

Par. equation 2 _____

Par. equation 1 _____

Par .equation 2 _____

Max height 1: _____   Range 1: _____   Max height 1: _____   Range 1: _____

Max height 2: _____   Range 2: _____   Max height 2: _____   Range 2: _____

Example 10) A baseball player is at bat and makes contact with the ball at a height of 3 ft. The ball leaves the bat at 110 miles per hour towards the center field fence, 425 feet away which is 12 feet high. If the ball leaves the bat at the following angles of elevation, determine whether or not the ball will be a home run. Show your equations and explain your answers.

a. $\theta = 17°$

b. $\theta = 18°$

Polar Coordinates

We are used to the coordinate system where coordinates are x and y. The parametric equations we just saw use a 3rd variable t, but still graphs points in the form of (x, y). Thus the system is called the rectangular system or sometimes referred to the Cartesian coordinate system. This system is based on straight lines – thus a rectangle. The polar coordinate system is based on a circle.

To form the polar coordinate system, we fix a point O called the pole or the origin. Each point P in the plane can be assigned polar coordinates (r, θ) as follows: r is the directed distance from O to P and θ is the directed angled, counterclockwise from polar axis to segment $\overline{OP}$. The diagram below shows three points on the polar coordinate system. It is convenient to locate points with respect to a grid of concentric circles intersected by radial lines through the pole. The line $\theta = 0$ is called the **polar axis.**
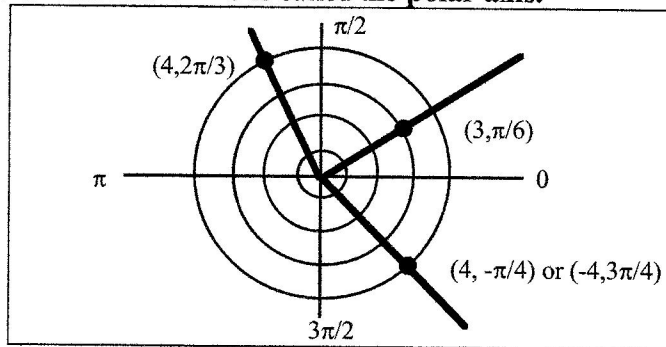
Depending on whether you want to measure angles as radians or degrees, the coordinate changes but still refers to the same point. Thus the point $\left(3, \dfrac{\pi}{6}\right)$ is the same point as $(3, 30°)$.

1997: BC-1

During the time period from $t = 0$ to $t = 6$ seconds, a particle moves along the path given by $x(t) = 3 \cos(\pi t)$ and $y(t) = 5 \sin(\pi t)$.

(a) Find the position of the particle when $t = 2.5$.

(b) On the axes provided below, sketch the graph of the path of the particle from $t = 0$ to $t = 6$. Indicate the direction of the particle along its path.

Note: The axes for this graph are provided in the pink test booklet only.

(c) How many times does the particle pass through the point found in part (a) ?

(d) Find the velocity vector for the particle at any time t.

(e) Write and evaluate an integral expression, in terms of sine and cosine, that gives the distance the particle travels from time $t = 1.25$ to $t = 1.75$.

Stuff You Must Know Cold – Pre-Cal

Even – Odd Functions	**Trig Graph**

Even – Odd Functions

If $f(x)$ is even, then

If $f(x)$ is odd, then

Composite Functions

$f(x) =$

$g(x) =$

$f(g(x)) =$

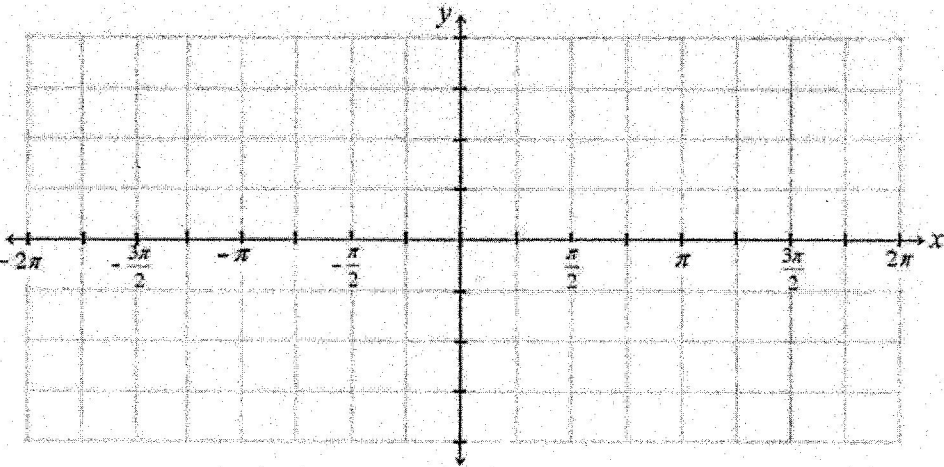**Triangles**

Law of Cosines:

Law of Sines:

Values of Trigonometric Functions for Common Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$			
$\dfrac{\pi}{6}$			
$\dfrac{\pi}{4}$			
$\dfrac{\pi}{3}$			
$\dfrac{\pi}{2}$			
π			

*Know both the **inverse trig** and the **trig** values. E.g.* $\tan^{-1}(1)$
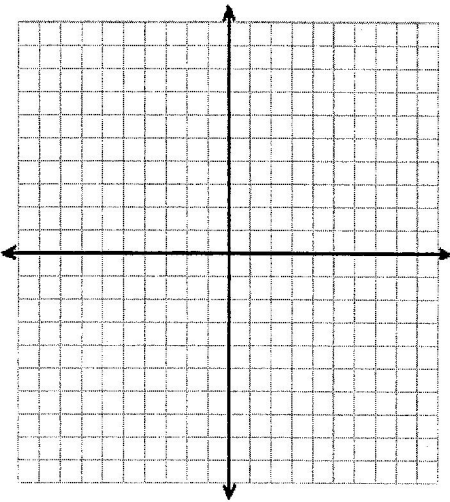
Trig Graph

$y =$

Parametric Equations

Graph
$x(t) =$

$y(t) =$

Polar Equations

Standard Formulas
$x =$

$y =$

$\tan\theta =$

$x^2 + y^2 =$

Graph
$r =$